Finite Majorizing Measures

D i s s e r t a t i o n

zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der Fakultät für

Mathematik und Informatik

der Friedrich–Schiller–Universität Jena

von Dipl.-Math. Bettina Bühler

geboren am 29.04.1972 in Dresden

Contents

Chapter 1

Introduction

1.1 Centered Gaussian Random Functions

A centered Gaussian random function (g.r.f.) indexed by T is a collection $X = (X_t)_{t \in T} = (X(t,\cdot))_{t \in T}$ of real valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that each finite linear combination

$$\sum_{\ell=1}^{n} a_\ell X_{t_\ell}\,;\ a_\ell \in \mathbb{R}, t_\ell \in T, n \in \mathbb{N},$$

is a centered real valued Gaussian variable, i.e. distributed according to $\mathcal{N}(0, \sigma_a^2)$ with $\sigma_a^2 = \langle Ra, a \rangle$ for $a = (a_1, \ldots, a_n)$ and $R = (\mathbb{E}X_{t_\ell}X_{t_k})_{\ell,k \le n}$.

Thus the distribution of the g.r.f. X is completely determined by its covariance structure $(\mathbb{E}X_sX_t)_{s,t \in T}$. It is therefore natural to analyze properties of g.r.f., which depend on the distribution only, by using the geometry of T for the induced $L_2(\Omega, \mathcal{F}, \mathbb{P})$-pseudo-metric $d_X(s,t) = (\mathbb{E}(X_s - X_t)^2)^{\frac{1}{2}}$. Interesting properties of this kind are the existence of a modification with bounded or with uniformly continuous sample paths $(X(\cdot,\omega) : T \to \mathbb{R})_{\omega \in \Omega}$. A collection $Y = (Y_t)_{t \in T}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a modification of X if $\mathbb{P}(X_t = Y_t) = 1$ for all $t \in T$.

We suppose that (T, d_X) is separable. Then the following condition is satisfied (see [F2] 3.1.3.):

There exists a modification of X with bounded sample paths iff

$$\sup_{\substack{S\subseteq T\\ \text{finite}}} \mathbb{E} \sup_{t\in S} X_t < \infty.$$

There exists a modification of X with uniformly continuous sample paths iff

$$\lim_{\varepsilon\to 0} \sup_{\substack{S\subseteq T\\ \text{finite}}} \mathbb{E} \sup_{\substack{d(s,t)<\varepsilon\\ s,t\in S}} |X_s - X_t| = 0.$$

To shorten the notion, we will write

$$\mathbb{E} \sup_{t\in T} X_t \qquad \text{instead of} \qquad \sup_{\substack{S\subseteq T\\ \text{finite}}} \mathbb{E} \sup_{t\in S} X_t.$$

The separability of (T, d_X) implies the separability of the induced subspace of $L_2(\Omega, \mathcal{F}, \mathbb{P})$, i.e. of

$$L_2 \text{ -closure of } \left\{ \sum_{\ell=1}^{n} a_\ell X_{t_\ell} \,;\ a_\ell \in \mathbb{R}, t_\ell \in T, n \in \mathbb{N} \right\}.$$

Thus we can find an orthonormal basis $g_1, g_2, \dots$ of this subspace. Of course, $g_1, g_2, \dots$ are i.i.d. $\sim \mathcal{N}(0,1)$ and $X_t = \sum_{\ell=1}^{\infty} (\mathbb{E}X_t g_\ell) g_\ell$ (convergence in L_2 and a.e.), $(\mathbb{E}X_t^2)^{\frac{1}{2}} = \left(\sum_{\ell=1}^{\infty} (\mathbb{E}X_t g_\ell)^2\right)^{\frac{1}{2}}$.

The situation can be simplified by assuming T being a subset of

$$\ell_2 = \left\{ t = (t_\ell)_{\ell=1}^{\infty} \,;\ t_\ell \in \mathbb{R}, \|t\|_2 = \left(\sum_{\ell=1}^{\infty} t_\ell^2 \right)^{\frac{1}{2}} < \infty \right\},$$

and

$$g_1, g_2, \dots \text{ being i.i.d. } \sim \mathcal{N}(0,1) \text{ and } X_t = \sum_{\ell=1}^{\infty} t_\ell g_\ell.$$

In this case the induced L_2-pseudo-metric is a metric, namely

$$d_X(s,t) = \|s - t\|_2,$$

and

$$X_t - X_s = X_{t-s} \quad \text{a. e..}$$

Write $T \in GB \quad (T \in GC)$ iff $X = (X_t)_{t\in T}$ possesses a modification with bounded (with bounded and uniformly continuous) sample paths.

1.2 Covering Numbers

Covering numbers are one classical possibility to investigate metric spaces. For example, characterization of GB and GC properties was first tried by using covering numbers.

For later use (chapter 2) let us introduce the notion of this numbers for an arbitrary metric space (T, d). We denote the closed ball $\{s \in T; d(s,t) \le \varepsilon\}$ by $B(t,\varepsilon)$. Then, for any subset A of T and $\varepsilon > 0$ the covering number $N(A, d, \varepsilon)$ is defined by

$$N(A,d,\varepsilon) = \inf\left\{ n \in \mathbb{N}; \bigvee_{t_1,\ldots,t_n \in T} \left(A \subseteq \bigcup_{\ell=1}^{n} B(t_\ell, \varepsilon)\right)\right\}.$$

Sometimes it is convenient to use a slightly different variant, namely

$$M(A,d,\varepsilon) = \sup\left\{ m \in \mathbb{N}; \bigvee_{s_1,\ldots,s_m \in A} \bigwedge_{\substack{1\le \ell,k\le m \\ \ell\ne k}} \big(d(s_\ell, s_k) > \varepsilon\big)\right\}.$$

We can now show that $N(A,d,\varepsilon) \le M(A,d,\varepsilon) \le N(A,d,\frac{\varepsilon}{2})$:
If $m = M(A,d,\varepsilon)$ then $A \subseteq \bigcup\limits_{\ell=1}^{m} B(s_\ell, \varepsilon)$, because otherwise we could find an element $s \in A \backslash \{s_1, \ldots, s_m\}$ with $d(s, s_\ell) > \varepsilon$ for $1 \le \ell \le m$.
Thus $N(A,d,\varepsilon) \le M(A,d,\varepsilon)$.
If $n = N(A,d,\frac{\varepsilon}{2})$ and $A \subseteq \bigcup\limits_{\ell=1}^{n} B(t_\ell, \frac{\varepsilon}{2})$ then we get the second statement using

$$\operatorname{card}\left(\{s_1,\ldots,s_m\} \cap B\left(t_\ell, \frac{\varepsilon}{2}\right)\right) \le 1 \text{ for } 1 \le \ell \le n.$$

In 1967 Dudley proved ([Du]) that

$$\int_0^\infty \sqrt{\ln N(T, d_X, \varepsilon)}\, d\varepsilon < \infty$$

is sufficient for $T \in GB$ and $T \in GC$. On the other hand Sudakov proved in 1973 ([Su]) that

$$\sup_{\varepsilon > 0} \varepsilon \sqrt{\ln N(T, d_X, \varepsilon)} < \infty$$

is necessary for $T \in GB$ and that the condition

$$\lim_{\varepsilon \to 0} \varepsilon \sqrt{\ln N(T, d_X, \varepsilon)} = 0$$

is necessary for $T \in GC$.

We believe that it is impossible to characterize GB and GC properties by using covering numbers $N(T, d_X, \varepsilon), \varepsilon > 0$. To prove this we should find two sets T_1 and T_2 satisfying $N(T_1, d_X, \varepsilon) = N(T_2, d_X, \varepsilon)$ for all $\varepsilon > 0$ and $T_1 \in GB, T_2 \notin GB$ or $T_1 \in GC, T_2 \notin GC$, respectively.

This seems to be difficult, because it is hard to compute all covering numbers exactly. However, the following two sets at least show that it is impossible to find a simple characterization using $(\ln N(T, d_X, \varepsilon))_{\varepsilon > 0}$ for the GB property:

Let

$$T_1 = \left\{ \frac{1}{\sqrt{\ln n}} e_n \, ; \; n = 2, 3, \ldots \right\},$$

where $e_n = (e_\ell^n)_{\ell=1}^\infty \in \ell_2$ with $e_n^n = 1$ and $e_\ell^n = 0$ for $n \neq \ell$ and let

$$T_2 = \left\{ t = (t_\ell)_{\ell=1}^\infty \in \ell_2 \, ; \; \sum_{\ell=1}^\infty \ell \cdot t_\ell^2 \leq 1 \right\}.$$

Then (see section 2.1.4)

$$N(T_1, d_X, 2^{-k}) = \operatorname{card} \left\{ \ell \geq 2 \, ; \; \frac{1}{\sqrt{\ln \ell}} \geq 2^{-k} \right\} + 1 = [\exp(2^{2k})].$$

We use [T2] (section 3) to get for T_2

$$N(T_2, d_X, 2^{-k}) \geq \exp\left(\text{card}\left\{\ell\,;\ \frac{1}{\sqrt{\ell}} \geq 2^{-k+1}\right\} \cdot \ln 2\right) = \exp(2^{2k-2} \cdot \ln 2)$$

and

$$\begin{aligned} N(T_2, d_X, 2^{-k}) &\leq \exp\left(c_0 \sum_{\ell=0}^{k+1} (k-\ell+4)\text{card}\left\{m\,;\ 2^{-\ell} \leq \frac{1}{\sqrt{m}} < 2^{-\ell+1}\right\}\right) \\ &= \exp\left(3c_0 \cdot 2^{2k} \sum_{j=0}^{k} (j+3)2^{-2j} + c_0(k+4)\right) \\ &\leq \exp(c \cdot 2^{2k}). \end{aligned}$$

Therefore, for suitable constants $c_1, c_2 > 0$ we obtain

$$c_1 \cdot 2^{2k} \leq \ln N(T_1, d_X, 2^{-k}), \ln N(T_2, d_X, 2^{-k}) \leq c_2 \cdot 2^{2k}.$$

On the other hand we know that $T_1 \in GB$ and $T_2 \notin GB$.

1.3 Majorizing Measures

Majorizing measures are a suitable tool to characterize the GB and GC properties. The sufficiency was proved by Fernique in 1975 ([F1]) and the necessity by Talagrand in 1987 ([T1]).

The advantage, compared with covering numbers, is that they take possible inhomogeneities of T into account.

For the GB property the characterization can be formulated as

$$c_1 \cdot \mathcal{I}\Theta(T, d_X) \leq \mathbb{E} \sup_{t \in T} X_t \leq c_2 \cdot \mathcal{I}\Theta(T, d_X)$$

for suitable constants $c_1, c_2 > 0$ and

$$\mathcal{I}\Theta(T, d_X) = \inf\left\{\sup_{t\in T} \int_0^\infty \sqrt{\ln \frac{1}{\mu(B(t,\varepsilon))}}\, d\varepsilon\,;\ \mu \in \mathcal{P}(T, d_X)\right\}.$$

We denote by $\mathcal{P}(T, d_X)$ the set of all probability measures on (T, d_X) with respect to the Borel σ-algebra.

From a technical point of view it is easier to work with $\Theta(T, d_X)$ instead of $\mathcal{I}\Theta(T, d_X)$. However, the definition of $\Theta(T, d_X)$ is more complicated, so that we will postpone it to section 2.1.1. $\Theta(T, d_X)$ is used by Talagrand. He also proved the equivalence between $\Theta(T, d_X)$ and $\mathcal{I}\Theta(T, d_X)$ ([T2]).

Another quantity was considered by Fernique ([F2]):

$$\sup \left\{ \int_0^\infty \int_T \sqrt{\ln \frac{1}{\nu(B(t,\varepsilon))}}\, \nu(dt)\, d\varepsilon \,;\; \nu \in \mathcal{P}(T, d_X) \right\}.$$

We will denote this quantity by , (T, d_X).
Fernique proved that , $(T, d_X) < \infty$ iff $T \in GB$ (5.4.1. in [F2]).

It is possible to consider majorizing measures and the relations between the different representations in a pure geometric way (without g.r.f.) on an arbitrary metric space (T, d). We will do so in chapter 2 for finite majorizing measures, i.e. for a family of quantities, indexed by N_1 and N_2, which are closely related to majorizing measures. Finite majorizing measures collect the information in a certain area of fineness. If (T, d) is precompact and $N_2 < \infty$, they are finite (therefore we chose their name). Majorizing measures in the usual sense are included for $N_1 = N_0, N_2 = \infty$, where N_0 will be defined in the beginning of chapter 2. In particular, we get a geometric proof for the equivalence between $\Theta(T, d_X)$ and , (T, d_X).

Finite majorizing measures with $N_2 < \infty$ can be constructed inductively (see section 2.5). For $N_2 = \infty$ this is not possible in a direct way, but by use of the result of section 2.4.3.

Chapter 2 may be helpful for everybody who would like to understand majorizing measures. In chapter 3 we consider relations between finite majorizing measures on (T, d_X) for $T \subseteq \ell_2$ and the g.r.f. $X = (X_t)_{t \in T}$.

We conclude our introduction with some notations:
Throughout the entire work, we denote by q a sufficiently large fixed number; for the definitions $q > 1$ will be large enough and for the theorems we will give rough possible bounds.

Furthermore we denote by c, c_0, c_1, c_2 positive constants which may depend on q. C may be changed in each occurance; c_0, c_1, c_2 remain fixed in each proof (this should be clear from context).

Acknowledgment

I would like to thank very much my academic advisor W. Linde who stimulated and supported this work.

Chapter 2

Finite Majorizing Measures on Metric Spaces

Let (T, d) be a metric space. We suppose that

$$0 < D(T) = \sup_{s,t \in T} d(s,t) < \infty.$$

As the case $D(T) = 0$ is not interesting and would need separate consideration within the following proofs, we will always assume $D(T) > 0$. In case $D(T) = \infty$ all finite majorizing measures would be infinite as can be seen by the Sudakov-type-estimations.

We define $N_0 \in \mathbb{Z}$ by

$$N(T, d, q^{-N_0}) = 1 < N(T, d, q^{-N_0 - 1}).$$

By $0 < D(T) < \infty$ we ensure the existence of N_0.

Furthermore, let $N_1 \in \mathbb{Z}, N_2 \in \mathbb{Z} \cup \{\infty\}$ be numbers with $N_1 < N_2$. Finally, we denote by $\mathcal{P}(T, d)$ the set of all probability measures on (T, d) with respect to the Borel σ-algebra.

2.1 $\Theta_{N_1}^{N_2}$

2.1.1 Definition of $\Theta_{N_1}^{N_2}$ and $M\Theta_{N_1}^{N_2}$

Let $\mathcal{Z}_{N_1}^{N_2,\not\subseteq} = \mathcal{Z}_{N_1}^{N_2,\not\subseteq}(T,d)$ be the set of all sequences

$$\mathcal{A} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N_2}) \qquad (\text{or } \mathcal{A} = (\mathcal{A}_{N_1}, \mathcal{A}_{N_1+1}, \ldots) \quad \text{for } N_2 = \infty)$$

satisfying the following conditions:

(a) $\mathcal{A}_j$ is a finite measurable partition of T, $N_1 \leq j \leq N_2$

(b) $N(A, d, q^{-j}) = 1$ for $A \in \mathcal{A}_j$, $N_1 \leq j \leq N_2$.

We denote the set of $\mathcal{A}_j$, which contains t, by $A_j(t)$.
Furthermore let $\mathcal{Z}_{N_1}^{N_2} = \mathcal{Z}_{N_1}^{N_2}(T,d)$ be the set of all increasing sequences of $\mathcal{Z}_{N_1}^{N_2,\not\subseteq}$, i.e. the set of sequences satisfying **(a), (b)** and

(c) $A_{j+1}(t) \subseteq A_j(t)$ for $t \in T$ and $N_1 \leq j < N_2$.

We call

$$\omega = (\omega_{N_1}, \ldots, \omega_{N_2}) \qquad (\text{or } \omega = (\omega_{N_1}, \omega_{N_1+1}, \ldots) \text{ for } N_2 = \infty)$$

a weight for $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2,\not\subseteq}$ (write $\omega \in \mathcal{G}(\mathcal{A})$) if

$$\omega_j : \mathcal{A}_j \to [0,\infty) \quad \text{with} \quad \sum_{A \in \mathcal{A}_j} \omega_j(A) \leq 1 \quad \text{for } N_1 \leq j \leq N_2.$$

Let

$$\Theta_{N_1,\mathcal{A},\omega}^{N_2} = \Theta_{N_1,\mathcal{A},\omega}^{N_2}(T,d) = \sup_{t \in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(t))}}$$

for $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2,\not\subseteq}, \omega \in \mathcal{G}(\mathcal{A})$. Then we are able to define a first representation of finite majorizing measures:

$$\Theta_{N_1}^{N_2} = \Theta_{N_1}^{N_2}(T,d) = \inf\{\Theta_{N_1,\mathcal{A},\omega}^{N_2}\,;\, \mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}, \omega \in \mathcal{G}(\mathcal{A})\}.$$

Sometimes it is useful to replace the weight ω by a probability measure. Let

$$M\Theta_{N_1,\mathcal{A},\mu}^{N_2} = M\Theta_{N_1,\mathcal{A},\mu}^{N_2}(T,d) = \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}}$$

for $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2,\not\subseteq}$, $\mu \in \mathcal{P}(T,d)$ and define

$$M\Theta_{N_1}^{N_2} = M\Theta_{N_1}^{N_2}(T,d) = \inf\{M\Theta_{N_1,\mathcal{A},\mu}^{N_2}\,;\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}, \mu \in \mathcal{P}(T,d)\}.$$

Note that $\mathcal{A}_{N_1}$ and ω_{N_1} do not influence $\Theta_{N_1}^{N_2}$ or $M\Theta_{N_1}^{N_2}$, so one could always take $\mathcal{A}_{N_1} = \mathcal{A}_{N_1+1}$ and $\omega_{N_1} = \omega_{N_1+1}$. We do not change the definition because the stated form is more convenient for Talagrand's construction.

It is sufficient to consider $N_1 \geq N_0$ because if $N_1 < N_0$ one could take $\mathcal{A}_j = \{T\}$ and $\omega_j(T) = 1$ for $j \leq N_0$ and thus

$$\Theta_{N_1}^{N_2} = \Theta_{N_0}^{N_2}\,, \quad M\Theta_{N_1}^{N_2} = M\Theta_{N_0}^{N_2} \quad \text{if } N_2 > N_0$$

and $\Theta_{N_1}^{N_2} = 0$, $M\Theta_{N_1}^{N_2} = 0$ if $N_2 \leq N_0$.

2.1.2 Equivalence of $\Theta_{N_1}^{N_2}$ and $M\Theta_{N_1}^{N_2}$

Proposition 2.1.2 $\Theta_{N_1}^{N_2} \leq M\Theta_{N_1}^{N_2} \leq c\Theta_{N_1}^{N_2}$.

Proof: For the first inequality let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\mu \in \mathcal{P}(T,d)$ be arbitrary. Set $\omega_j(A) = \mu(A)$ for $A \in \mathcal{A}_j$ and $N_1 \leq j \leq N_2$.
Of course $\omega = (\omega_{N_1}, \ldots, \omega_{N_2}) \in \mathcal{G}(\mathcal{A})$, and we obtain

$$\begin{aligned}\Theta_{N_1,\mathcal{A},\omega}^{N_2} &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(t))}} \\ &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}} \\ &= M\Theta_{N_1,\mathcal{A},\mu}^{N_2}.\end{aligned}$$

For the second inequality let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$ be arbitrary.

We choose some point $t_A \in A$ for any $A \in \mathcal{A}_j$ and $N_1 \le j \le N_2$ and set

$$\mu_0 = \sum_{j=N_1+1}^{N_2} 2^{-j+N_1} \sum_{A \in \mathcal{A}_j} \omega_j(A) \cdot \delta_{t_A}.$$

We get

$$\mu_0(T) = \sum_{j=N_1+1}^{N_2} 2^{-j+N_1} \sum_{A \in \mathcal{A}_j} \omega_j(A) \le \sum_{j=N_1+1}^{N_2} 2^{-j+N_1} \le 1.$$

Thus there exists a $\mu \in \mathcal{P}(T,d)$ with $\mu \ge \mu_0$.

We obtain

$$\begin{aligned} M\Theta_{N_1,\mathcal{A},\mu}^{N_2} &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}} \\ &\le \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{2^{-j+N_1}\omega_j(A_j(t))}} \\ &\le \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{j-N_1}\sqrt{\ln 2} \\ &\quad + \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(t))}} \\ &\le q^{-N_1} \left(\sum_{j=1}^{\infty} q^{-j}\sqrt{j} \right) \sqrt{\ln 2} + \Theta_{N_1,\mathcal{A},\omega}^{N_2}. \end{aligned}$$

It is sufficient to consider $N_1 \ge N_0$ (see section 2.1.1), therefore

$$\mathrm{card}\mathcal{A}_{N_1+1} \ge 2,$$

and we can find an $A \in \mathcal{A}_{N_1+1}$ with $\omega_{N_1+1}(A) \le \frac{1}{2}$, so that

$$\Theta_{N_1,\mathcal{A},\omega}^{N_2} \ge q^{-(N_1+1)}\sqrt{\ln 2}$$

and

$$M\Theta_{N_1,\mathcal{A},\mu}^{N_2} \le \left(q \sum_{j=1}^{\infty} q^{-j}\sqrt{j} + 1 \right) \Theta_{N_1,\mathcal{A},\omega}^{N_2}.$$

□

2.1.3 Sudakov-Dudley-Bounds

We call the following estimations Sudakov-Dudley-bounds to remember their results for g.r.f. (see section 1.2). The proof for our geometric quantities is easier, however.

Proposition 2.1.3

$$\sup_{N_1+1\le j\le N_2} q^{-j}\sqrt{\ln N(T,d,q^{-j})} \le \Theta_{N_1}^{N_2} \le c\int_{q^{-N_2-1}}^{q^{-N_1-1}} \sqrt{\ln N(T,d,\varepsilon)}\,d\varepsilon$$

Proof: For the left hand side let $\mathcal{A}\in\mathcal{Z}_{N_1}^{N_2}$ and $\omega\in\mathcal{G}(\mathcal{A})$ be arbitrary. Using $N(A,d,q^{-j})=1$ for $A\in\mathcal{A}_j$ and $N_1\le j\le N_2$, we find elements $t(A)\in T$ with

$$T\subseteq\bigcup_{A\in\mathcal{A}_j} A\subseteq\bigcup_{A\in\mathcal{A}_j} B(t(A),q^{-j}).$$

Thus $N(T,d,q^{-j})\le \mathrm{card}\mathcal{A}_j$, and there exists a set $A\in\mathcal{A}_j$ for each j with

$$\omega_j(A)\le\frac{1}{\mathrm{card}\mathcal{A}_j}\le\frac{1}{N(T,d,q^{-j})}.$$

Fix $t\in A$. Then

$$\Theta_{N_1,\mathcal{A},\omega}^{N_2}\ge q^{-j}\sqrt{\ln\frac{1}{\omega_j(A_j(t))}}\ge q^{-j}\sqrt{\ln N(T,d,q^{-j})},$$

provided that $N_1+1\le j\le N_2$.
For the right hand side we will construct a suitable $\mathcal{A}\in\mathcal{Z}_{N_1}^{N_2},\omega\in\mathcal{G}(\mathcal{A})$ as follows:
For $N_1+1\le j\le N_2$ we choose a partition $\mathcal{B}_j$ of T which is induced by a covering of T with $N(T,d,q^{-j})$ balls of radius q^{-j}. Let

$$\mathcal{A}_{N_1}=\mathcal{A}_{N_1+1}=\mathcal{B}_{N_1+1}.$$

For $N_1+1\le j<N_2$ we set

$$\mathcal{A}_{j+1}=\{A\cap B;\ A\cap B\ne\emptyset, A\in\mathcal{A}_j, B\in\mathcal{B}_{j+1}\}.$$

Obviously, we get for $N_1 + 1 \leq j \leq N_2$

$$\mathrm{card}\mathcal{A}_j \leq \prod_{\ell=N_1+1}^{j} N(T, d, q^{-\ell}).$$

Therefore, it is possible to set

$$\omega_j(A) = \left(\prod_{\ell=N_1+1}^{j} N(T, d, q^{-\ell}) \right)^{-1} \quad \text{for } A \in \mathcal{A}_j \quad (\text{and } \omega_{N_1} = \omega_{N_1+1}).$$

We obtain

$$\begin{aligned}
\sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(t))}} &= \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \prod_{\ell=N_1+1}^{j} N(T, d, q^{-\ell})} \\
&\leq \sum_{j=N_1+1}^{N_2} \sum_{\ell=N_1+1}^{j} q^{-j} \sqrt{\ln N(T, d, q^{-\ell})} \\
&\leq \frac{q}{q-1} \sum_{\ell=N_1+1}^{N_2} q^{-\ell} \sqrt{\ln N(T, d, q^{-\ell})} \\
&\leq \left(\frac{q}{q-1} \right)^2 \int_{q^{-N_2-1}}^{q^{-N_1-1}} \sqrt{\ln N(T, d, \varepsilon)} \, d\varepsilon.
\end{aligned}$$

□

2.1.4 Example (Sequence)

The following example shows how the Sudakov-bound can be achieved (up to a constant). We will give an explicit construction (see [LL]) to demonstrate what can be done with the definitions. In section 2.5.5, we consider the example again to deliver a better understanding.

Let $T = \mathbb{N}$ and let $1 > a_1 > a_2 > \ldots$ be a sequence approaching zero and

$$d(k, \ell) = \begin{cases} \sqrt{a_k^2 + a_\ell^2} & ; \quad k \neq \ell \\ 0 & ; \quad k = \ell. \end{cases}$$

Note that we obtain this metric space from the g.r.f.

$$X = (X_k)_{k\in\mathbb{N}},\ X_k = a_k g_k,\ g_1, g_2, \ldots \text{ i.i.d. } \sim \mathcal{N}(0,1).$$

Set $\sigma(\varepsilon) = \mathrm{card}\{k; a_k \geq \varepsilon\}$ so that $a_1 > \ldots > a_{\sigma(\varepsilon)} \geq \varepsilon > a_{\sigma(\varepsilon)+1} > \ldots$.

Using this notation we get $N(T,d,\varepsilon) = \sigma(\varepsilon)+1$, because it is possible to cover T by $B(k,\varepsilon) = \{k\}, k \leq \sigma(\varepsilon)$, and $B(\ell,\varepsilon) = \{\sigma(\varepsilon)+1, \sigma(\varepsilon)+2, \ldots\}$ for a (large) ℓ with $a_\ell^2 \leq \varepsilon^2 - a_{\sigma(\varepsilon)+1}^2$.

It is clear that we can not take fewer balls of radius ε, because $k, k \leq \sigma(\varepsilon)$, is only contained in $B(k,\varepsilon)$. Therefore we have to take all $B(k,\varepsilon), k \leq \sigma(\varepsilon)$, and we need at least one additional ball to cover $\{\sigma(\varepsilon)+1, \sigma(\varepsilon)+2, \ldots\}$.
Set $\sigma_n = \sigma(q^{-n})$.
We consider $N_1 < N_2$ with $\sigma_{N_1} \geq 1$, i.e. with $a_1 \geq q^{-N_1}$, and denote by K the quantity

$$q \cdot \sup_{N_1 < j \leq N_2} q^{-j}\sqrt{\ln \sigma_j} = q \cdot \sup_{N_1 < j \leq N_2} q^{-j}\sqrt{\ln(N(T,d,q^{-j})-1)}.$$

We are now able to define suitable $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$:
Set $\mathcal{A}_{N_1} = \mathcal{A}_{N_1+1}, \omega_{N_1} = \omega_{N_1+1}$ and for $N_1 < j \leq N_2$

$$\mathcal{A}_j = \{\{1\}, \ldots, \{\sigma_j\}, B_j\} \qquad \text{with } B_j = \{\sigma_j+1, \sigma_j+2, \ldots\}.$$

Then $\{k\} = B(k, q^{-j})$ for $k \leq \sigma_j$ and $B_j = B(k_j, q^{-j})$, where k_j is chosen with $a_{k_j}^2 \leq q^{-2j} - a_{\sigma_j+1}^2$, hence $\mathcal{A} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N_2}) \in \mathcal{Z}_{N_1}^{N_2}$.

While the choice of the partitions is straightforward, suitable weights are not that obvious. We take $\omega_j(B_j) = \frac{1}{3}$ and

$$\omega_j(\{k\}) = \begin{cases} \frac{1}{3\sigma_{N_1}} & ; \quad 1 \leq k \leq \sigma_{N_1} \\ a_k \exp(\frac{-K^2}{a_k^2}) \cdot \frac{q-1}{3q} \cdot q^{N_1} & ; \quad \sigma_{N_1} < k \leq \sigma_j. \end{cases}$$

Now we have to show that $\omega \in \mathcal{G}(\mathcal{A})$, i.e.

$$\sum_{k=1}^{\sigma_j} \omega_j(\{k\}) + \omega_j(B_j) \leq 1 \quad \text{ for } N_1+1 \leq j \leq N_2:$$

$$\begin{aligned}
&\sum_{k=1}^{\sigma_j} \omega_j(\{k\}) + \omega_j(B_j) \\
\leq\ & \sum_{k=1}^{\sigma_{N_1}} \frac{1}{3\sigma_{N_1}} + \sum_{k=\sigma_{N_1}+1}^{\sigma_{N_2}} a_k \exp\left(\frac{-K^2}{a_k^2}\right) \cdot \frac{q-1}{3q} \cdot q^{N_1} + \frac{1}{3} \\
=\ & \frac{2}{3} + \frac{q-1}{3q} \cdot q^{N_1} \sum_{n=N_1+1}^{N_2} \sum_{q^{-n} \leq a_k < q^{-n+1}} a_k \exp\left(\frac{-K^2}{a_k^2}\right) \\
\leq\ & \frac{2}{3} + \frac{q-1}{3q} \cdot q^{N_1} \sum_{n=N_1+1}^{N_2} q^{-n+1} \exp(-K^2 q^{2n-2}) \sigma_n \\
=\ & \frac{2}{3} + \frac{q-1}{3q} \cdot q^{N_1} \sum_{n=N_1+1}^{N_2} q^{-n+1} \exp\left(q^{2n-2}\left[-K^2 + \left(q \cdot q^{-n}\sqrt{\ln \sigma_n}\right)^2\right]\right) \\
\leq\ & \frac{2}{3} + \frac{q-1}{3q} \cdot q^{N_1} \sum_{n=N_1+1}^{N_2} q^{-n+1} \\
\leq\ & 1,
\end{aligned}$$

by definition of K. It remains to estimate $\Theta_{N_1,\mathcal{A},\omega}^{N_2}$ in a manner that shows the desired result. For $k \leq \sigma_{N_1}$ we get

$$\sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(k))}} = \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln 3\sigma_{N_1}} \leq c q^{-N_1} \left(\sqrt{\ln 3} + \sqrt{\ln \sigma_{N_1}}\right).$$

For $k > \sigma_{N_1}$, i.e. for $a_k < q^{-N_1}$, we obtain

$$\begin{aligned}
&\sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(k))}} \\
=\ & \sum_{\substack{N_1+1 \leq j \leq N_2 \\ \sigma_j \geq k}} q^{-j} \sqrt{\ln \frac{\exp\left(\frac{K^2}{a_k^2}\right) \cdot 3q}{a_k (q-1) q^{N_1}}} + \sum_{\substack{N_1+1 \leq j \leq N_2 \\ \sigma_j < k}} q^{-j} \sqrt{\ln 3} \\
\leq\ & \left(\sum_{\substack{N_1+1 \leq j \leq N_2 \\ q^{-j} \leq a_k}} q^{-j}\right) \left(\frac{K}{a_k} + \sqrt{\ln \frac{3q}{q-1}} + \sqrt{\ln \frac{1}{a_k q^{N_1}}}\right) + c q^{-N_1} \\
=\ & (*).
\end{aligned}$$

We now choose j_0 with $q^{-j_0} \le a_k < q^{-j_0+1}$ and get

$$\sum_{\substack{N_1+1 \le j \le N_2 \\ q^{-j} \le a_k}} q^{-j} = \sum_{j=j_0}^{N_2} q^{-j} \le c q^{-j_0} \le c a_k.$$

Hence we are able to continue with

$$\begin{aligned} (*) \quad &\le c\left(K + a_k + a_k\sqrt{\ln \frac{1}{a_k q^{N_1}}} + q^{-N_1}\right) \\ &\le c\left(K + q^{-N_1} + q^{-N_1}\left[a_k q^{N_1}\sqrt{\ln \frac{1}{a_k q^{N_1}}}\right]\right) \\ &\le c(K + q^{-N_1}), \end{aligned}$$

where we have used that $x \mapsto x\sqrt{\ln \frac{1}{x}}$ is bounded on $(0,1]$.
By using $\sigma_{N_1} \le \sigma_{N_1+1}$ and $N(T,d,q^{-N_1-1}) = \sigma_{N_1+1} + 1 \ge 2$,
we obtain altogether

$$\begin{aligned} \Theta^{N_2}_{N_1,\mathcal{A},\omega} &= \sup_{k\in\mathbb{N}} \sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln \frac{1}{\omega_j(A_j(k))}} \\ &\le c\left(K + q^{-N_1} + q^{-N_1}\sqrt{\ln \sigma_{N_1}}\right) \\ &\le c\left(K + q^{-(N_1+1)}\sqrt{\ln 2} + q^{-(N_1+1)}\sqrt{\ln \sigma_{N_1+1}}\right) \\ &\le c \sup_{N_1<j\le N_2} q^{-j}\sqrt{\ln N(T,d,q^{-j})}. \end{aligned}$$

2.2 $\Gamma^{N_2}_{N_1}$

2.2.1 Definition of $\Gamma^{N_2}_{N_1}$ and $F\Gamma^{N_2}_{N_1}$

We set for $\nu \in \mathcal{P}(T,d)$

$$, {}^{N_2}_{N_1,\nu} = , {}^{N_2}_{N_1,\nu}(T,d) = \int_{2q^{-N_2}}^{2q^{-N_1}} \int_T \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\right)}}\, \nu(dt)\, d\varepsilon.$$

It is not clear (at least if (T,d) is not separable) whether the function under the second integral is measurable. If not, the integral should be understood as outer integral (see section 1.2 in [VW]), i.e. as

$$\inf\left\{ \int_T h(t)\,\nu(dt)\,;\ h(t) \geq \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\right)}},\ h \text{ measurable } \right\}.$$

We can now define a second representation of finite majorizing measures:

$$,\, {}^{N_2}_{N_1} = \,,\, {}^{N_2}_{N_1}(T,d) = \sup\{,\, {}^{N_2}_{N_1,\nu}; \nu \in \mathcal{P}(T,d)\}.$$

Furthermore it is useful to consider the following variant of $,\, {}^{N_2}_{N_1}$:

$$F,\, {}^{N_2}_{N_1} = F,\, {}^{N_2}_{N_1}(T,d) = \sup\{,\, {}^{N_2}_{N_1,\nu}; \nu \in \mathcal{P}(T,d), \operatorname{card}\operatorname{supp}\nu < \infty\}.$$

One advantage of $F,\, {}^{N_2}_{N_1}$ is that we do not have any problems with measurability. Obviously, $F,\, {}^{N_2}_{N_1} \leq \,,\, {}^{N_2}_{N_1}$.

It is sufficient to consider $N_1 \geq N_0$ because if $N_1 < N_0$ then $\nu(B(t,\varepsilon)) = 1$ for $\varepsilon > 2q^{-N_0}$ and thus

$$,\, {}^{N_2}_{N_1} = \,,\, {}^{N_2}_{N_0},\ F,\, {}^{N_2}_{N_1} = F,\, {}^{N_2}_{N_0} \quad \text{for } N_2 > N_0$$

and $,\, {}^{N_2}_{N_1} = 0$, $F,\, {}^{N_2}_{N_1} = 0$ for $N_2 \leq N_0$.

2.2.2 Elementary Properties

Proposition 2.2.2

(a) $F,\, {}^{N_2}_{N_1}(T,d) = \sup\{,\, {}^{N_2}_{N_1}(S, d_{|S\times S});\ S \subseteq T, 1 < \operatorname{card} S < \infty\}$

(b) $S \subseteq T \Longrightarrow F,\, {}^{N_2}_{N_1}(S, d_{|S\times S}) \leq F,\, {}^{N_2}_{N_1}(T,d)$

Proof:

(a) We get

$$\begin{aligned} & F,\, {}^{N_2}_{N_1}(T,d) \\ = \ & \sup\{,\, {}^{N_2}_{N_1,\nu}(T,d);\ \nu \in \mathcal{P}(T,d), \operatorname{card}\operatorname{supp}\nu < \infty\} \end{aligned}$$

$$
\begin{aligned}
&= \sup\left\{ ,\ {}_{N_1,\nu}^{N_2}(T,d);\ \begin{array}{l} \nu \in \mathcal{P}(T,d), \mathrm{supp}\nu \subseteq S, \\ S \subseteq T, \mathrm{card} S < \infty \end{array} \right\} \\
&= \sup\left\{ \int_{2q^{-N_2}}^{2q^{-N_1}} \int_T \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\right)}}\, \nu(dt)\, d\varepsilon;\ \begin{array}{l} \nu \in \mathcal{P}(T,d), \mathrm{supp}\nu \subseteq S, \\ S \subseteq T, \mathrm{card} S < \infty \end{array} \right\} \\
&= \sup\left\{ \int_{2q^{-N_2}}^{2q^{-N_1}} \int_S \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\cap S\right)}}\, \nu(dt)\, d\varepsilon;\ \begin{array}{l} \nu \in \mathcal{P}(S,d_{|S\times S}), \\ S \subseteq T, \mathrm{card} S < \infty \end{array} \right\} \\
&= \sup\{ ,\ {}_{N_1,\nu}^{N_2}(S, d_{|S\times S});\ \nu \in \mathcal{P}(S,d_{|S\times S}), S \subseteq T, \mathrm{card} S < \infty\}.
\end{aligned}
$$

Observe that $,\ {}_{N_1,\nu}^{N_2}(S, d_{|S\times S}) = 0$ for $S \subseteq T$ with $\mathrm{card} S = 1$. Furthermore, there exists some $S \subseteq T$ with $1 < \mathrm{card} S < \infty$, because we supposed that $D(T) > 0$. Hence we obtain the stated result.

(b) This follows directly from **(a)**. $\square$

2.2.3 Sudakov-Dudley-Bounds

Proposition 2.2.3 Let $q > 2$, then

$$
c \sup_{N_1 \le j \le N_2 - 1} q^{-j} \sqrt{\ln M(T,d,q^{-j})} \le F,\ {}_{N_1}^{N_2} \le ,\ {}_{N_1}^{N_2} \le 2 \int_{q^{-N_2}}^{q^{-N_1}} \sqrt{\ln N(T,d,\varepsilon)}\, d\varepsilon.
$$

Proof: For the lower bound we choose points $s_1, \ldots, s_m$ for $N_1 \le j \le N_2 - 1$ with $m = M(T,d,q^{-j})$ and $d(s_\ell, s_k) > q^{-j}, 1 \le \ell, k \le m, \ell \ne k$. Using $\nu = \frac{1}{m}\sum_{\ell=1}^{m} \delta_{s_\ell}$ we obtain

$$
\begin{aligned}
F,\ {}_{N_1}^{N_2}(T,d) &\ge \int_{2q^{-N_2}}^{2q^{-N_1}} \frac{1}{m} \sum_{\ell=1}^{m} \sqrt{\ln \frac{1}{\nu\left(B(s_\ell,\varepsilon)\right)}}\, d\varepsilon \\
&\ge \int_{2q^{-j-1}}^{q^{-j}} \sqrt{\ln m}\, d\varepsilon
\end{aligned}
$$

$$= \left(1-\frac{2}{q}\right) q^{-j}\sqrt{\ln M(T,d,q^{-j})}.$$

For the upper bound it is sufficient to show for $\nu \in \mathcal{P}(T,d)$

$$\int_T \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\right)}}\,\nu(dt) \le \sqrt{\ln N\left(T,d,\frac{\varepsilon}{2}\right)}.$$

For proof we take a partition $\mathcal{A}_\varepsilon$ of T which is induced by a cover of T using $N\left(T,d,\frac{\varepsilon}{2}\right)$ balls with radius $\frac{\varepsilon}{2}$. We denote the set of $\mathcal{A}_\varepsilon$, which contains t, by $A_\varepsilon(t)$. Of course, $A_\varepsilon(t) \subseteq B(t,\varepsilon)$ and therefore

$$\int_T \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\right)}}\,\nu(dt) \le \sum_{A\in\mathcal{A}_\varepsilon} \sqrt{\ln \frac{1}{\nu(A)}}\cdot \nu(A).$$

We now consider the function $f(x) = x\sqrt{\ln \frac{1}{x}}$, $0 < x \le 1$. This is a concave function because

$$f''(x) = -\frac{1}{2x\sqrt{\ln \frac{1}{x}}} - \frac{1}{4x\left(\ln \frac{1}{x}\right)^{\frac{3}{2}}}.$$

Hence

$$f\left(\sum_{\ell=1}^{n} \frac{1}{n} x_\ell\right) \ge \sum_{\ell=1}^{n} \frac{1}{n} f(x_\ell),$$

i.e.

$$\sum_{\ell=1}^{n} x_\ell \sqrt{\ln \frac{1}{x_\ell}} \le \sum_{\ell=1}^{n} x_\ell \sqrt{\ln \frac{n}{\sum\limits_{k=1}^{n} x_k}}.$$

Thus we obtain with $\sum\limits_{A\in\mathcal{A}_\varepsilon} \nu(A) = 1$

$$\sum_{A\in\mathcal{A}_\varepsilon} \sqrt{\ln \frac{1}{\nu(A)}}\cdot \nu(A) \le \sqrt{\ln N\left(T,d,\frac{\varepsilon}{2}\right)},$$

as desired. $\square$

2.2.4 Example (Group)

Let $(T, +)$ be a compact Abelian group and let d be a metric satisfying

$$\bigwedge_{r,s,t \in T} \big(d(s+r, t+r) = d(s,t) \big).$$

Hence there exists the Haar measure ν.

Obviously,

$$M(T, d, 2\varepsilon) \cdot \nu(B(0, \varepsilon)) \leq 1.$$

Thus we obtain

$$\begin{aligned} ,^{N_2}_{N_1}(T,d) &\geq \; ,^{N_2}_{N_1,\nu}(T,d) \\ &= \int_{2q^{-N_2}}^{2q^{-N_1}} \int_T \sqrt{\ln \frac{1}{\nu\left(B(t,\varepsilon)\right)}}\, \nu(dt)\, d\varepsilon \\ &= \int_{2q^{-N_2}}^{2q^{-N_1}} \sqrt{\ln \frac{1}{\nu\left(B(0,\varepsilon)\right)}}\, d\varepsilon \\ &\geq \int_{2q^{-N_2}}^{2q^{-N_1}} \sqrt{\ln M(T,d,2\varepsilon)}\, d\varepsilon \\ &\geq \frac{1}{2} \int_{4q^{-N_2}}^{4q^{-N_1}} \sqrt{\ln N(T,d,\varepsilon)}\, d\varepsilon, \end{aligned}$$

i.e. $,^{N_2}_{N_1}(T,d)$ almost achieves the Dudley-bound. A concrete example is

$$T = \{\sigma = (\sigma_j)_{j=1}^{\infty}; \sigma_j \in \{-1, 1\}\}$$

with

$$d(\sigma, \sigma') = 2 \left(\sum_{\substack{1 \leq \ell < \infty \\ \sigma_\ell \neq \sigma'_\ell}} a_\ell^2 \right)^{1/2}$$

and $a = (a_\ell)_{\ell=1}^{\infty} \in \ell_2$ fixed.

We obtain this metric space from the g.r.f. $X_\sigma = \sum_{\ell=1}^{\infty} \sigma_\ell a_\ell g_\ell$ with $g_1, g_2, \ldots$ i.i.d. $\sim \mathcal{N}(0,1)$.

2.3 Talagrand's Construction

2.3.1 Initial Situation

We denote the set of all partitions $\mathcal{A}_{N_1}$ which are induced by a cover of T using $N(T,d,q^{-N_1})$ or $N(T,d,q^{-N_1})+1$ balls (the second case will be useful in section 3.4) of radius q^{-N_1} by $SP(N_1)$. We write $\omega_{N_1} \in S\mathcal{G}(\mathcal{A}_{N_1})$ if $\omega_{N_1}(A) = (N(T,d,q^{-N_1})+1)^{-1}$ for $A \in \mathcal{A}_{N_1}$.

If $\mathcal{A}_{N_1} \in SP(N_1)$ then we can find (not necessary uniquely determined) points $u(A) \in T$ with $A \subseteq B(u(A), q^{-N_1})$ for $A \in \mathcal{A}_{N_1}$.

We then denote functions $u : T \to T$ satisfying $u(t) = u(A)$ for $t \in A$ by $SF(\mathcal{A}_{N_1})$, and we set $SF(N_1) = \{u; u \in SF(\mathcal{A}_{N_1}), \mathcal{A}_{N_1} \in SP(N_1)\}$.

Proposition 2.3.1 Let $N_1 \geq N_0, \widetilde{\mathcal{A}_{N_1}} \in SP(N_1)$ and $\widetilde{\omega_{N_1}} \in S\mathcal{G}(\widetilde{\mathcal{A}_{N_1}})$. Then

$$\Theta_{N_1}^{N_2} \leq \inf\{\Theta_{N_1,\mathcal{A},\omega}^{N_2}; \mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}, \omega \in \mathcal{G}(\mathcal{A}), \mathcal{A}_{N_1} = \widetilde{\mathcal{A}_{N_1}}, \omega_{N_1} = \widetilde{\omega_{N_1}}\} \leq c\Theta_{N_1}^{N_2}.$$

Proof: The lower bound is obvious by the definition of $\Theta_{N_1}^{N_2}$. For the upper bound let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$ be arbitrary. We set

$$\widetilde{\mathcal{A}_j} = \{A \cap B; A \cap B \neq \emptyset, A \in \mathcal{A}_j, B \in \widetilde{\mathcal{A}_{N_1}}\}$$

for $N_1 < j \leq N_2$ and

$$\widetilde{\omega_j}(A \cap B) = \omega_j(A) \cdot \big(N(T,d,q^{-N_1})+1\big)^{-1} \qquad \text{for } A \in \mathcal{A}_j, B \in \widetilde{\mathcal{A}_{N_1}}.$$

Obviously, $\widetilde{\mathcal{A}} = (\widetilde{\mathcal{A}_{N_1}}, \ldots, \widetilde{\mathcal{A}_{N_2}}) \in \mathcal{Z}_{N_1}^{N_2}$ and $\widetilde{\omega} = (\widetilde{\omega_{N_1}}, \ldots, \widetilde{\omega_{N_2}}) \in \mathcal{G}(\widetilde{\mathcal{A}})$.

Furthermore we obtain

$$
\begin{aligned}
&\Theta^{N_2}_{N_1,\widetilde{\mathcal{A}},\widetilde{\omega}}\\
= \; & \sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{\widetilde{\omega}_j(\widetilde{A_j}(t))}}\\
= \; & \sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{N(T,d,q^{-N_1})+1}{\omega_j(A_j(t))}}\\
\leq \; & \sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{\omega_j(A_j(t))}}+\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln(N(T,d,q^{-N_1})+1)}\\
\leq \; & \Theta^{N_2}_{N_1,\mathcal{A},\omega}+q^{-N_1-1}\frac{q}{q-1}\sqrt{\ln(2\cdot N(T,d,q^{-N_1}))}.
\end{aligned}
$$

Using proposition 2.1.3 and $N(T,d,q^{-N_1-1})\geq 2$ for $N_1\geq N_0$ we obtain

$$q^{-N_1-1}\sqrt{\ln(2\cdot N(T,d,q^{-N_1}))}\leq q^{-N_1-1}\sqrt{\ln(N(T,d,q^{-N_1-1}))^2}\leq\sqrt{2}\Theta^{N_2}_{N_1}.$$

Hence

$$\Theta^{N_2}_{N_1,\widetilde{\mathcal{A}},\widetilde{\omega}}\leq\left(1+\sqrt{2}\frac{q}{q-1}\right)\Theta^{N_2}_{N_1,\mathcal{A},\omega}.$$

□

2.3.2 Proof of Talagrand's Construction

Theorem 2.3.2 Let $N_1\geq N_0$ and $\phi_{N_1},\dots,\phi_{N_2+1}:T\to[0,\infty)$ be functions satisfying the following property:

For $N_1\leq j\leq N_2-1$, $t\in T$ and any points $t_1,\dots,t_n\in B(t,q^{-j})$ satisfying $d(t_\ell,t_k)>q^{-j-1}, 1\leq \ell,k\leq n,\ell\neq k$ it holds

$$\phi_j(t)\geq c_0q^{-j}\sqrt{\ln n}+\min_{1\leq\ell\leq n}\phi_{j+2}(t_\ell).$$

Denote $\sup\{\phi_j(t);\ t\in T,N_1\leq j\leq N_2+1\}$ by $\phi^{N_2}_{N_1}$. Then we obtain

$$\Theta^{N_2}_{N_1}\leq c\left(\phi^{N_2}_{N_1}+q^{-N_1}+q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}\right).$$

Proof: For $\phi_{N_1}^{N_2} = \infty$ it is nothing to show. So we assume $\phi_{N_1}^{N_2} < \infty$. We start our construction with (an arbitrary chosen)

$$\mathcal{A}_{N_1} \in SP(N_1), \omega_{N_1} \in S\mathcal{G}(\mathcal{A}_{N_1}) \text{ and } u_{N_1} \in SF(\mathcal{A}_{N_1}).$$

Having $\mathcal{A}_j, \omega_j$ and u_j with $A \subseteq B(u_j(A), q^{-j})$ for $A \in \mathcal{A}_j$ we construct $\mathcal{A}_{j+1}, \omega_{j+1}$ and u_{j+1} $(N_1 \leq j \leq N_2 - 1)$. To get $\mathcal{A}_{j+1}$ it is sufficient (and necessary) to take apart any $A \in \mathcal{A}_j$. For that purpose we choose $t_1 \in A$ with

$$\phi_{j+2}(t_1) \geq \sup\{\phi_{j+2}(t); t \in A\} - 2^{N_1-j-1}\phi_{N_1}^{N_2}$$

and we set $D_1 = A \cap B(t_1, q^{-j-1})$.

Inductively we take $t_k \in A \setminus \bigcup_{\ell<k} D_\ell$ with

$$\phi_{j+2}(t_k) \geq \sup\left\{\phi_{j+2}(t);\ t \in A \setminus \bigcup_{\ell<k} D_\ell\right\} - 2^{N_1-j-1}\phi_{N_1}^{N_2}$$

and we set

$$D_k = \left(A \setminus \bigcup_{\ell<k} D_\ell\right) \cap B(t_k, q^{-j-1}).$$

Thus we obtain a measurable partition $\{D_1, D_2, \ldots\}$ of A which is finite, as $t_1, t_2, \ldots$ are elements of $A \subseteq B(u_j(A), q^{-j})$ satisfying $d(t_\ell, t_k) > q^{-j-1}$ for $\ell \neq k$ and hence

$$c_0 q^{-j}\sqrt{\ln \operatorname{card}\{D_1, D_2, \ldots\}} \leq \phi_j(u_j(A)) \leq \phi_{N_1}^{N_2} < \infty.$$

Setting $u_{j+1}(D_k) = t_k$ and $\omega_{j+1}(D_k) = \frac{\omega_j(A)}{4k^2}$ we finally obtain $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$, the latter because of

$$\begin{aligned}
\sum_{D\in\mathcal{A}_{j+1}} \omega_{j+1}(D) &= \sum_{A\in\mathcal{A}_j} \sum_{\substack{D\in\mathcal{A}_{j+1} \\ D\subseteq A}} \omega_{j+1}(D) \\
&\leq \sum_{A\in\mathcal{A}_j} \omega_j(A) \sum_{k=1}^{\infty} \frac{1}{4k^2} \\
&\leq \sum_{A\in\mathcal{A}_j} \omega_j(A).
\end{aligned}$$

It remains to show the desired inequality. By construction and assumption we get

$$\phi_j(u_j(A)) \geq c_0 q^{-j}\sqrt{\ln k} + \min_{1\leq \ell \leq k} \phi_{j+2}(t_\ell)$$

for $N_1 \leq j \leq N_2 - 1$ and $k \leq \operatorname{card}\{D_1, D_2, \ldots\}$.
Furthermore we obtain for $\ell \leq k$ from the definition of t_ℓ:

$$\phi_{j+2}(t_\ell) \geq \phi_{j+2}(t_k) - 2^{N_1-j-1}\phi_{N_1}^{N_2}$$

and thus

$$\phi_j(u_j(A)) \geq c_0 q^{-j}\sqrt{\ln k} + \phi_{j+2}(t_k) - 2^{N_1-j-1}\phi_{N_1}^{N_2}.$$

For $t \in D_k$ we have $A = A_j(t), D_k = A_{j+1}(t)$ and (by definition of ω_{j+1}) $k = \sqrt{\frac{\omega_j(A_j(t))}{4\omega_{j+1}(A_{j+1}(t))}}$. Hence

$$\phi_j\left(u_j(A_j(t))\right) \geq c_0 q^{-j}\sqrt{\ln\sqrt{\frac{\omega_j\left(A_j(t)\right)}{4\omega_{j+1}\left(A_{j+1}(t)\right)}}} + \phi_{j+2}(t_k) - 2^{N_1-j-1}\phi_{N_1}^{N_2}.$$

By construction we get $u_j(A) \in A$ for $j \geq N_1 + 1$, in particular

$$u_{j+2}(A_{j+2}(t)) \in A_{j+2}(t) \subseteq A_{j+1}(t) = D_k.$$

Thus

$$\phi_{j+2}(t_k) \geq \phi_{j+2}\left(u_{j+2}(A_{j+2}(t))\right) - 2^{N_1-j-1}\phi_{N_1}^{N_2}$$

for $N_1 \leq j \leq N_2 - 2$ and $\phi_{j+2}(t_k) \geq 0$ for $j = N_2 - 1$.
Altogether we obtain

$$\begin{aligned}\sum_{j=N_1}^{N_2-1}\phi_j\left(u_j(A_j(t))\right) \quad &\geq \quad \sum_{j=N_1}^{N_2-2}\phi_{j+2}\left(u_{j+2}(A_{j+2}(t))\right) - \sum_{j=N_1}^{N_2-1} 2^{N_1-j}\phi_{N_1}^{N_2} \\ &\quad + c_0\sum_{j=N_1}^{N_2-1} q^{-j}\sqrt{\frac{1}{2}\ln\frac{\omega_j\left(A_j(t)\right)}{4\omega_{j+1}\left(A_{j+1}(t)\right)}},\end{aligned}$$

hence

$$
\begin{aligned}
& \frac{c_0}{\sqrt{2}} \sum_{j=N_1}^{N_2-1} q^{-j} \left(\sqrt{\ln \frac{1}{\omega_{j+1}\left(A_{j+1}(t)\right)}} - \sqrt{\ln \frac{1}{\omega_j\left(A_j(t)\right)}} - \sqrt{\ln 4} \right) \\
\leq \;& \frac{c_0}{\sqrt{2}} \sum_{j=N_1}^{N_2-1} q^{-j} \sqrt{\ln \frac{\omega_j\left(A_j(t)\right)}{4\omega_{j+1}\left(A_{j+1}(t)\right)}} \\
\leq \;& 2\phi^{N_2}_{N_1} + \phi_{N_1}\left(u_{N_1}(A_{N_1}(t))\right) + \phi_{N_1+1}\left(u_{N_1+1}(A_{N_1+1}(t))\right) \\
\leq \;& 4\phi^{N_2}_{N_1}.
\end{aligned}
$$

From this we get

$$
\begin{aligned}
& \frac{c_0}{\sqrt{2}} \sum_{j=N_1+1}^{N_2-1} (q^{-j+1} - q^{-j}) \sqrt{\ln \frac{1}{\omega_j\left(A_j(t)\right)}} + \frac{c_0}{\sqrt{2}} q^{-N_2+1} \sqrt{\ln \frac{1}{\omega_{N_2}\left(A_{N_2}(t)\right)}} \\
\leq \;& 4\phi^{N_2}_{N_1} + \frac{c_0}{\sqrt{2}} \sum_{j=N_1}^{N_2-1} q^{-j} \sqrt{\ln 4} + \frac{c_0}{\sqrt{2}} q^{-N_1} \sqrt{\ln \frac{1}{\omega_{N_1}\left(A_{N_1}(t)\right)}}.
\end{aligned}
$$

Using $\omega_{N_1} \in S\mathcal{G}(\mathcal{A}_{N_1})$ and

$$
\sqrt{\ln(N(T,d,q^{-N_1})+1)} \leq \sqrt{\ln 2} + \sqrt{\ln N(T,d,q^{-N_1})}
$$

we obtain

$$
\sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j(A_j(t))}} \leq c\left(\phi^{N_2}_{N_1} + q^{-N_1} + q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}\right).
$$

□

2.4 Relations between $\Gamma^{N_2}_{N_1}$ and $\Theta^{N_2}_{N_1}$

2.4.1 Lower Bound

Theorem 2.4.1 Let $N_1 \geq N_0$ and $q > 8$. Then

$$
\Theta^{N_2}_{N_1} \leq c \cdot F,\ {}^{N_2+1}_{N_1}.
$$

Proof: Set for $N_1 \leq j \leq N_2, t \in T$

$$\phi_j(t) = \sup\{,\,^{N_2+1}_{j,\nu}(T,d); \nu \in \mathcal{P}(T,d), \mathrm{supp}\nu \subseteq B(t, 2q^{-j}), \mathrm{card}\,\mathrm{supp}\nu < \infty\}$$

and $\phi_{N_2+1}(t) = 0, t \in T$.
Obviously,

$$\sup\{\phi_j(t); t \in T, N_1 \leq j \leq N_2+1\} \leq F,\,^{N_2+1}_{N_1}(T,d).$$

Let $N_1 \leq j < N_2$, $t \in T$ and $t_1, \dots, t_n \in B(t, q^{-j})$ with $d(t_\ell, t_k) > q^{-j-1}$, $\ell, k \leq n, \ell \neq k$ be given. Fix $\delta > 0$ and choose $\nu_1, \dots, \nu_n \in \mathcal{P}(T,d)$ satisfying $\mathrm{card}\,\mathrm{supp}\nu_\ell < \infty, \mathrm{supp}\nu_\ell \subseteq B(t_\ell, 2q^{-j-2})$ and

$$\phi_{j+2}(t_\ell) - \delta < ,\,^{N_2+1}_{j+2,\nu_\ell}(T,d), 1 \leq \ell \leq n.$$

Set $\nu = \frac{1}{n}\sum\limits_{\ell=1}^{n} \nu_\ell$.
Obviously, $\nu \in \mathcal{P}(T,d), \mathrm{card}\,\mathrm{supp}\nu < \infty$ and $\mathrm{supp}\nu \subseteq B(t, 2q^{-j})$, the latter because of $\bigcup\limits_{\ell=1}^{n} B(t_\ell, 2q^{-j-2}) \subseteq B(t, 2q^{-j})$.
Hence we obtain

$$\begin{aligned}
\phi_j(t) &\geq ,\,^{N_2+1}_{j,\nu}(T,d) \\
&= \int\limits_{2q^{-N_2-1}}^{2q^{-j}} \int\limits_T \sqrt{\ln \frac{1}{\nu\left(B(s,\varepsilon)\right)}}\, \nu(ds)\, d\varepsilon \\
&= \int\limits_{2q^{-N_2-1}}^{2q^{-j}} \frac{1}{n}\sum_{\ell=1}^{n} \int\limits_{B(t_\ell, 2q^{-j-2})} \sqrt{\ln \frac{1}{\frac{1}{n}\sum\limits_{k=1}^{n} \nu_k(B(s,\varepsilon))}}\, \nu_\ell(ds)\, d\varepsilon.
\end{aligned}$$

If $\varepsilon + 4q^{-j-2} < q^{-j-1}$, $s \in B(t_\ell, 2q^{-j-2})$ and $\ell \neq k$ then

$$B(s,\varepsilon) \cap B(t_k, 2q^{-j-2}) = \emptyset$$

because $B(s,\varepsilon) \subseteq B(t_\ell, \varepsilon + 2q^{-j-2})$ and $d(t_\ell, t_k) > q^{-j-1}$.

Hence

$$\frac{1}{n}\sum_{k=1}^{n}\nu_k(B(s,\varepsilon)) = \frac{1}{n}\nu_\ell(B(s,\varepsilon))$$

for $s \in B(t_\ell, 2q^{-j-2})$ and $\varepsilon < \left(1-\frac{4}{q}\right)q^{-j-1}$.

Therefore we get

$$\phi_j(t) \geq \int_{2q^{-N_2-1}}^{\left(1-\frac{4}{q}\right)q^{-j-1}} \frac{1}{n}\sum_{\ell=1}^{n} \int_{B(t_\ell,2q^{-j-2})} \sqrt{\ln\frac{n}{\nu_\ell(B(s,\varepsilon))}}\,\nu_\ell(ds)\,d\varepsilon.$$

Furthermore, if $\varepsilon > 4q^{-j-2}$ then $B(s,\varepsilon) \supseteq B(t_\ell, 2q^{-j-2})$ for $s \in B(t_\ell, 2q^{-j-2})$.

Thus

$$\begin{aligned}
\phi_j(t) &\geq \frac{1}{n}\sum_{\ell=1}^{n} \int_{2q^{-N_2-1}}^{2q^{-j-2}} \int_{B(t_\ell,2q^{-j-2})} \sqrt{\ln\frac{n}{\nu_\ell(B(s,\varepsilon))}}\,\nu_\ell(ds)\,d\varepsilon \\
&\quad + \frac{1}{n}\sum_{\ell=1}^{n} \int_{4q^{-j-2}}^{\left(1-\frac{4}{q}\right)q^{-j-1}} \int_{B(t_\ell,2q^{-j-2})} \sqrt{\ln n}\,\nu_\ell(ds)\,d\varepsilon \\
&\geq \frac{1}{n}\sum_{\ell=1}^{n} \int_{2q^{-N_2-1}}^{2q^{-j-2}} \int_{T} \sqrt{\ln\frac{1}{\nu_\ell(B(s,\varepsilon))}}\,\nu_\ell(ds)\,d\varepsilon + \left(1-\frac{8}{q}\right)q^{-j-1}\sqrt{\ln n} \\
&> \frac{1}{n}\sum_{\ell=1}^{n}(\phi_{j+2}(t_\ell)-\delta) + \left(1-\frac{8}{q}\right)q^{-j-1}\sqrt{\ln n}.
\end{aligned}$$

Letting $\delta \to 0$ we obtain

$$\begin{aligned}
\phi_j(t) &\geq \frac{1}{n}\sum_{\ell=1}^{n}\phi_{j+2}(t_\ell) + \left(1-\frac{8}{q}\right)q^{-j-1}\sqrt{\ln n} \\
&\geq \min_{1\leq\ell\leq n}\phi_{j+2}(t_\ell) + \left(1-\frac{8}{q}\right)q^{-j-1}\sqrt{\ln n}.
\end{aligned}$$

It follows from Talagrand's construction that

$$\Theta_{N_1}^{N_2} \leq c\left(F,\Gamma_{N_1}^{N_2+1} + q^{-N_1} + q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}\right).$$

By proposition 2.2.3 we get

$$F, {}^{N_2+1}_{N_1} \geq cq^{-N_1}\sqrt{\ln M(T,d,q^{-N_1})} \geq cq^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}$$

and by $N_1 < N_2$ and $N_1 \geq N_0$

$$F, {}^{N_2+1}_{N_1} \geq cq^{-N_1-1}\sqrt{\ln M(T,d,q^{-N_1-1})} \geq cq^{-N_1-1}\sqrt{\ln 2}.$$

Thus $\Theta^{N_2}_{N_1} \leq c \cdot F, {}^{N_2+1}_{N_1}$. □

2.4.2 Upper Bound

Lemma 2.4.2 Let f be an increasing function from $[1,\infty)$ to $[0,\infty)$ satisfying $f(1) = 0$, $\lim\limits_{x\to\infty} f(x) = \infty$ and

$$\bigvee_{c_0 \geq 1} \bigwedge_{x,y} \big(f(xy) \leq c_0(f(x)+f(y))\big).$$

Then there exist constants c_1 and c_2 such that

$$\begin{aligned} xf\left(\frac{1}{x}\right) &\leq c_1 f(2) \quad \text{for} \quad 0 < x \leq 1 \text{ and} \\ xf\left(\frac{1}{x}\right) &\leq c_2(x+y)f\left(\frac{1}{y}\right) \quad \text{for} \quad 0 < x \leq 1 \text{ and } 0 < y \leq \frac{1}{2}. \end{aligned}$$

Proof: For $\frac{1}{2} < x \leq 1$ we have $xf\left(\frac{1}{x}\right) \leq xf(2) \leq f(2)$. Thus, choosing a natural number $m \geq 1$ with $x \in (2^{-2^m}, 2^{-2^{m-1}}]$ for $x \leq \frac{1}{2}$ we conclude

$$xf\left(\frac{1}{x}\right) \leq \frac{1}{2^{2^{m-1}}} f(2^{2^m}) \leq \frac{(2c_0)^m f(2)}{2^{2^{m-1}}}.$$

Hence we may take $c_1 = \max\left\{\frac{(2c_0)^m}{2^{2^{m-1}}}\,; m = 1,2,\ldots\right\}$.

For the second inequality choose $n \geq 1$ with $y \in (2^{-2^n}, 2^{-2^{n-1}}]$.

If $y \leq x$, then there is nothing to prove. Otherwise we have $n \leq m$ with m as above. If $m = n$, then

$$f\left(\frac{1}{x}\right) \leq f(2^{2^n}) \leq 2c_0 f(2^{2^{n-1}}) \leq 2c_0 f\left(\frac{1}{y}\right),$$

hence $xf\left(\frac{1}{x}\right) \leq 2c_0 xf\left(\frac{1}{y}\right)$. For $m > n$ we get

$$f\left(\frac{1}{x}\right) \leq f(2^{2^{n-1}\cdot 2^{\ell+1}}) \leq (2c_0)^{\ell+1} f(2^{2^{n-1}}) \leq (2c_0)^{\ell+1} f\left(\frac{1}{y}\right)$$

with $\ell = m - n \geq 1$, and therefore

$$xf\left(\frac{1}{x}\right) \leq \frac{(2c_0)^{\ell+1}}{2^{2^n(2^{\ell-1}-1)}} yf\left(\frac{1}{y}\right),$$

which yields the proposition with $c_2 = \max\left\{\frac{(2c_0)^{\ell+1}}{2^{(2^{\ell-1}-1)}}\,;\,\ell = 1, 2, \ldots\right\}$. $\square$

Theorem 2.4.2 Let $N_1 \geq N_0$. Then

$$,_{N_1}^{N_2} \leq c \cdot M\Theta_{N_1}^{N_2}.$$

Proof: (compare 5.3.7. in [F2])

Fix $\delta > 0$ and take $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}, \mu \in \mathcal{P}(T, d)$ with

$$M\Theta_{N_1}^{N_2} + \delta > M\Theta_{N_1,\mathcal{A},\mu}^{N_2} = \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{\mu(A_j(t))}}.$$

For any $\nu \in \mathcal{P}(T, d)$ we get by using $A_j(t) \subseteq B(t, 2q^{-j})$

$$\begin{aligned}
,_{N_1,\nu}^{N_2} &= \sum_{j=N_1+1}^{N_2} \int_{2q^{-j}}^{2q^{-j+1}} \int_T \sqrt{\ln\frac{1}{\nu\left(B(t,\varepsilon)\right)}}\,\nu(dt)\,d\varepsilon \\
&\leq \sum_{j=N_1+1}^{N_2} (2q-2)q^{-j} \int_T \sqrt{\ln\frac{1}{\nu(A_j(t))}}\,\nu(dt) \\
&= (2q-2)\sum_{j=N_1+1}^{N_2} q^{-j} \sum_{\substack{A\in\mathcal{A}_j \\ \nu(A)\neq 0}} \sqrt{\ln\frac{1}{\nu(A)}}\cdot\nu(A).
\end{aligned}$$

There exists at most one set $A \in \mathcal{A}_j$ with $\mu(A) > \frac{1}{2}$.

So, applying lemma 2.4.2 for $f(x) = \sqrt{\ln x}$, we obtain

$$
\begin{aligned}
, _{N_1,\nu}^{N_2} \quad \leq \quad & c_2(2q-2) \sum_{j=N_1+1}^{N_2} q^{-j} \sum_{A \in \mathcal{A}_j} \sqrt{\ln \frac{1}{\mu(A)}} \big(\nu(A) + \mu(A)\big) \\
& + c_1(2q-2) \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln 2} \\
= \quad & c_2(2q-2) \int_T \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}} \, (\mu+\nu)(dt) \\
& + c_1(2q-2) \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln 2} \\
\leq \quad & 4(q-1)c_2 \sup_{t \in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}} \\
& + 2(q-1)c_1 \sup_{t \in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}}.
\end{aligned}
$$

In the last line, we applied card$\mathcal{A}_{N_1+1} \geq 2$ for $N_1 \geq N_0$, so that we could find some $t \in T$ with $\mu(A_j(t)) \leq \mu(A_{N_1+1}(t)) \leq \frac{1}{2}, N_1 + 1 \leq j \leq N_2$.

Thus

$$, _{N_1,\nu}^{N_2} < \big(4(q-1)c_2 + 2(q-1)c_1\big)(M\Theta_{N_1}^{N_2} + \delta).$$

By $\delta \to 0$ we obtain the desired result. □

2.4.3 Equivalence of $\Theta_{N_1}^{\infty}$ and $\sup\{\Theta_{N_1}^{N_2}; N_2 < \infty\}$

Theorem 2.4.3 Let $N_1 \geq N_0$ and $q > 8$. Then

$$\sup\{\Theta_{N_1}^{N_2}; N_2 < \infty\} \leq \Theta_{N_1}^{\infty} \leq c \cdot \sup\{\Theta_{N_1}^{N_2}; N_2 < \infty\}.$$

Proof: The lower bound is easy to see:

If $\mathcal{A} \in \mathcal{Z}_{N_1}^{\infty}$ and $\omega \in \mathcal{G}(\mathcal{A})$ then

$$\widetilde{\mathcal{A}} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N_2}) \in \mathcal{Z}_{N_1}^{N_2} \text{ and } \widetilde{\omega} = (\omega_{N_1}, \ldots, \omega_{N_2}) \in \mathcal{G}(\widetilde{\mathcal{A}}).$$

Furthermore we get by definition

$$\Theta_{N_1}^{N_2} \leq \Theta_{N_1,\widetilde{\mathcal{A}},\widetilde{\omega}}^{N_1} \leq \Theta_{N_1,\mathcal{A},\omega}^{\infty}$$

and thus

$$\sup\{\Theta_{N_1}^{N_2}; N_2 < \infty\} \leq \Theta_{N_1}^{\infty}.$$

To prove the upper bound we first show that

$$\Theta_{N_1}^{\infty}(S, d_{|S\times S}) \leq c \cdot \sup\{\Theta_{N_1}^{N_2}(S, d_{|S\times S}); N_2 < \infty\}$$

for any $S \subseteq T$ with $1 < \mathrm{card} S < \infty$.

For such S we can find $N \in \mathbb{Z}, N > N_1$ with $d(s_1, s_2) > 2q^{-N}$ for all points $s_1, s_2 \in S, s_1 \neq s_2$.

If $\mathcal{A} \in \mathcal{Z}_{N_1}^N(S, d_{|S\times S})$ and $\omega \in \mathcal{G}(\mathcal{A})$, we set

$$\widetilde{\mathcal{A}} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N-1}, \mathcal{A}_N, \mathcal{A}_N, \ldots)$$

and

$$\widetilde{\omega} = (\omega_{N_1}, \ldots, \omega_{N-1}, \omega_N, \omega_N, \ldots).$$

It is clear that $\widetilde{\mathcal{A}} \in \mathcal{Z}_{N_1}^{\infty}(S, d_{|S\times S})$ and $\widetilde{\omega} \in \mathcal{G}(\widetilde{\mathcal{A}})$, because any $A \in \mathcal{A}_N$ satisfies $D(A) \leq 2q^{-N}$, and thus by our choice of N we have $\mathcal{A}_N = \{\{s\}; s \in S\}$.

We obtain

$$\begin{aligned}
& \Theta_{N_1,\widetilde{\mathcal{A}},\widetilde{\omega}}^{\infty}(S, d_{|S\times S}) \\
= \; & \sup_{s\in S}\left(\sum_{j=N_1+1}^{N-1} q^{-j}\sqrt{\ln\frac{1}{\omega_j(A_j(t))}} + \sum_{j=N}^{\infty} q^{-j}\sqrt{\ln\frac{1}{\omega_N(A_N(t))}}\right) \\
= \; & \sup_{s\in S}\left(\sum_{j=N_1+1}^{N-1} q^{-j}\sqrt{\ln\frac{1}{\omega_j(A_j(t))}} + \frac{q}{q-1}q^{-N}\sqrt{\ln\frac{1}{\omega_N(A_N(t))}}\right) \\
\leq \; & \frac{q}{q-1}\Theta_{N_1,\mathcal{A},\omega}^{N}(S, d_{|S\times S}).
\end{aligned}$$

Thus

$$\begin{aligned}
\Theta^{\infty}_{N_1}(S, d_{|S\times S}) &\leq \frac{q}{q-1}\Theta^{N}_{N_1}(S, d_{|S\times S}) \\
&\leq \frac{q}{q-1}\sup\{\Theta^{N_2}_{N_1}(S, d_{|S\times S}); N_2 < \infty\}.
\end{aligned}$$

We are now able to prove the desired result:

Using theorem 2.4.1 (here we use $q > 8$), proposition 2.2.2, theorem 2.4.2 and proposition 2.1.2 we get

$$\begin{aligned}
\Theta^{\infty}_{N_1}(T, d) &\leq c \cdot F,^{\infty}_{N_1}(T, d) \\
&\leq c \cdot \sup\{,^{\infty}_{N_1}(S, d_{|S\times S});\ S \subseteq T, 1 < \mathrm{card}S < \infty\} \\
&\leq c \cdot \sup\{M\Theta^{\infty}_{N_1}(S, d_{|S\times S});\ S \subseteq T, 1 < \mathrm{card}S < \infty\} \\
&\leq c \cdot \sup\{\Theta^{\infty}_{N_1}(S, d_{|S\times S});\ S \subseteq T, 1 < \mathrm{card}S < \infty\} \\
&\leq c \cdot \sup\{\sup_{N_2<\infty}\Theta^{N_2}_{N_1}(S, d_{|S\times S});\ S \subseteq T, 1 < \mathrm{card}S < \infty\}.
\end{aligned}$$

Furthermore we have $\Theta^{N_2}_{N_1}(S, d_{|S\times S}) \leq c \cdot \Theta^{N_2+1}_{N_1}(T, d)$ for $S \subseteq T$. This can be seen directly from the definitions (be careful: $N(A, d, q^{-j}) = 1$ does not imply $N(A \cap S, d_{|S\times S}, q^{-j}) = 1$). Alternatively, we may use the same theorems and propositions as in the last step:

$$\begin{aligned}
\Theta^{N_2}_{N_1}(S, d_{|S\times S}) &\leq c \cdot F,^{N_2+1}_{N_1}(S, d_{|S\times S}) \\
&\leq c \cdot F,^{N_2+1}_{N_1}(T, d) \\
&\leq c \cdot ,^{N_2+1}_{N_1}(T, d) \\
&\leq c \cdot M\Theta^{N_2+1}_{N_1}(T, d) \\
&\leq c \cdot \Theta^{N_2+1}_{N_1}(T, d).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\Theta^{\infty}_{N_1}(T, d) &\leq c \cdot \sup\left\{\sup_{N_2<\infty}\Theta^{N_2+1}_{N_1}(T, d);\ S \subseteq T, 1 < \mathrm{card}S < \infty\right\} \\
&\leq c \cdot \sup\{\Theta^{N_2}_{N_1}(T, d); N_2 < \infty\}.
\end{aligned}$$

□

2.5 Inductive Construction for $N_2 < \infty$

2.5.1 Definition of $\overline{\phi}_{N_1}^{N_2}$

Recall Talagrand's construction. In this section we will define a further quantity by looking at its assumption from an other point of view. A direct way is only possible for $N_2 < \infty$. We suppose $N_0 \leq N_1 < N_2$ through the entire section 2.5.

Set $\overline{\phi}_{N_2}(t) = \overline{\phi}_{N_2+1}(t) = 0$ for $t \in T$ and set inductively

$$\overline{\phi}_j(t) = \sup \left\{ c_0 q^{-j} \sqrt{\ln n} + \min_{1 \leq \ell \leq n} \overline{\phi}_{j+2}(t_\ell)\,; \begin{array}{l} n \in \mathbb{N}, t_1, \ldots, t_n \in B(t, q^{-j}), \\ d(t_\ell, t_k) > q^{-j-1}, \ell, k \leq n, \ell \neq k \end{array} \right\}$$

for $N_1 \leq j \leq N_2 - 1$.

Then we define $\overline{\phi}_{N_1}^{N_2}$ by

$$\overline{\phi}_{N_1}^{N_2} = \overline{\phi}_{N_1}^{N_2}(T, d) = \sup\{\overline{\phi}_j(t)\,; t \in T, N_1 \leq j \leq N_2 + 1\}.$$

Denote the constant of this definition by c_0 through the entire section 2.5.

2.5.2 Sudakov-Dudley-Bounds

We believe that it is instructive to prove the Sudakov-Dudley-bounds directly by means of the definition of $\overline{\phi}_{N_1}^{N_2}$.

Proposition 2.5.2

$$\begin{aligned} c_1 \sup_{N_1 \leq j \leq N_2} q^{-j} \sqrt{\ln N(T, d, q^{-j})} &\leq \overline{\phi}_{N_1}^{N_2} + q^{-N_1} \sqrt{\ln N(T, d, q^{-N_1})} \\ &\leq c_2 \int_{\frac{1}{2} q^{-N_2 - 1}}^{\frac{1}{2} q^{-N_1 - 1}} \sqrt{\ln N(T, d, \varepsilon)}\, d\varepsilon. \end{aligned}$$

Proof: We will prove the lower bound for $c_1 = \frac{c_0}{\max\left\{c_0, \frac{1}{(q-1)}\right\}}$.

First, from the definition of $\overline{\phi}_{N_1}^{N_2}$ follows that for $t \in T$ and $N_1 \le j < N_2$

$$\overline{\phi}_{N_1}^{N_2} \ge \overline{\phi}_j(t) \ge c_0 q^{-j} \sqrt{\ln M(B(t,q^{-j}),d,q^{-j-1})}.$$

Setting $n_j = \sup_{t\in T} N(B(t,q^{-j}),d,q^{-j-1})$ we obtain

$$\overline{\phi}_{N_1}^{N_2} \ge c_0 q^{-j} \sqrt{\ln n_j}$$

for $N_1 \le j < N_2$. Obviously,

$$N(T,d,q^{-j}) \le N(T,d,q^{-N_1}) \cdot n_{N_1} \cdot n_{N_1+1} \cdot \ldots \cdot n_{j-1}.$$

Thus we get for $N_1 \le j \le N_2$

$$\begin{aligned}
c_0 q^{-j}\sqrt{\ln N(T,d,q^{-j})} &\le c_0 q^{-j}\sqrt{\ln\left(N(T,d,q^{-N_1}) \cdot \prod_{\ell=N_1}^{j-1} n_\ell\right)} \\
&\le c_0 q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})} + \sum_{\ell=N_1}^{j-1} q^{-j+\ell} c_0 q^{-\ell}\sqrt{\ln n_\ell} \\
&\le c_0 q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})} + \frac{1}{q-1}\overline{\phi}_{N_1}^{N_2}
\end{aligned}$$

as desired. We now show the upper bound. Set

$$m_j = \sup_{t\in T} M(B(t,q^{-j}),d,q^{-j-1}).$$

First we prove inductively

$$\overline{\phi}_j(t) \le c_0 \sum_{\substack{j\le \ell\le N_2-1 \\ \ell\equiv j \bmod 2}} q^{-\ell}\sqrt{\ln m_\ell}$$

for $N_1 \le j \le N_2 - 1$.

By definition we get

$$\begin{aligned}
\overline{\phi}_{N_2-1}(t) &= c_0 q^{-(N_2-1)}\sqrt{\ln M(B(t,q^{-(N_2-1)}),d,q^{-N_2})} \\
&\le c_0 q^{-(N_2-1)}\sqrt{\ln m_{N_2-1}}
\end{aligned}$$

and

$$\begin{aligned}\overline{\phi}_{N_2-2}(t) &= c_0 q^{-(N_2-2)}\sqrt{\ln M(B(t,q^{-(N_2-2)}),d,q^{-(N_2-1)})}\\ &\le c_0 q^{-(N_2-2)}\sqrt{\ln m_{N_2-2}}.\end{aligned}$$

Moreover, it is clear that

$$\overline{\phi}_j(t) \le c_0 q^{-j}\sqrt{\ln M(B(t,q^{-j}),d,q^{-j-1})} + \sup_{s\in T}\overline{\phi}_{j+2}(s).$$

Thus

$$\begin{aligned}\overline{\phi}_j(t) &\le c_0 q^{-j}\sqrt{\ln m_j} + c_0 \sum_{\substack{j+2\le \ell\le N_2-1\\ \ell\equiv(j+2)\,\mathrm{mod}\,2}} q^{-\ell}\sqrt{\ln m_\ell}\\ &\le c_0 \sum_{\substack{j\le \ell\le N_2-1\\ \ell\equiv j\,\mathrm{mod}\,2}} q^{-\ell}\sqrt{\ln m_\ell}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\overline{\phi}_{N_1}^{N_2}(T,d) &\le c_0\sum_{\ell=N_1}^{N_2-1} q^{-\ell}\sqrt{\ln m_\ell}\\ &\le c_0\sum_{\ell=N_1}^{N_2-1} q^{-\ell}\sqrt{\ln M(T,d,q^{-\ell-1})}\\ &\le c_0\sum_{\ell=N_1}^{N_2-1} q^{-\ell}\sqrt{\ln N\left(T,d,\frac{q^{-\ell-1}}{2}\right)}\\ &\le \frac{2c_0q^2}{q-1}\int_{\frac12 q^{-N_2-1}}^{\frac12 q^{-N_1-1}}\sqrt{\ln N(T,d,\varepsilon)}\,d\varepsilon.\end{aligned}$$

Along with

$$\begin{aligned}\int_{\frac12 q^{-N_2-1}}^{\frac12 q^{-N_1-1}}\sqrt{\ln N(T,d,\varepsilon)}\,d\varepsilon &\ge \frac12 q^{-N_1-1}\left(1-\frac1q\right)\sqrt{\ln N\left(T,d,\frac{q^{-N_1-1}}{2}\right)}\\ &\ge \frac{q-1}{2q^2}\cdot q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}\end{aligned}$$

we get the result with $c_2 = \frac{2q^2}{q-1}(1+c_0)$. $\square$

2.5.3 Relations between $\overline{\phi}_{N_1}^{N_2}$ and $\Theta_{N_1}^{N_2}$

Theorem 2.5.3 For $q > 8$ it holds

$$c_1 \Theta_{N_1}^{N_2} \le \overline{\phi}_{N_1}^{N_2} + q^{-N_1}\sqrt{\ln N(T, d, q^{-N_1})} \le c_2 \Theta_{N_1}^{N_2+1}.$$

Proof: For the lower bound we apply theorem 2.3.2, which yields

$$\Theta_{N_1}^{N_2} \le c\left(\overline{\phi}_{N_1}^{N_2} + q^{-N_1} + q^{-N_1}\sqrt{\ln N(T, d, q^{-N_1})}\right).$$

Now we use proposition 2.5.2 to get

$$\overline{\phi}_{N_1}^{N_2} + q^{-N_1}\sqrt{\ln N(T, d, q^{-N_1})} \ge c \cdot q^{-N_1-1}\sqrt{\ln N(T, d, q^{-N_1-1})} \ge \frac{c}{q} q^{-N_1}\sqrt{\ln 2}.$$

Thus we obtain the desired inequality.

For the upper bound recall the proof of theorem 2.4.1. There we constructed (using $q > 8$) a family $\phi_{N_1}, \dots, \phi_{N_2+1}$ satisfying both the assumption of Talagrand's construction with constant $\frac{q-8}{q^2}$ and

$$\phi_{N_1}^{N_2} = \sup\{\phi_j(t); t \in T, N_1 \le j \le N_2 + 1\} \le F, {}_{N_1}^{N_2+1}.$$

Note that for any family of functions

$$\widetilde{\phi}_{N_1}, \dots, \widetilde{\phi}_{N_2+1} : T \to [0, \infty)$$

satisfying the assumption of Talagrand's construction with constant $\widetilde{c}_0$, we obtain

$$\frac{c_0}{\widetilde{c}_0}\widetilde{\phi}_j(t) \ge \overline{\phi}_j(t), \; t \in T, N_1 \le j \le N_2 + 1,$$

and hence $\frac{c_0}{\widetilde{c}_0}\widetilde{\phi_{N_1}^{N_2}} \ge \overline{\phi}_{N_1}^{N_2}$.

This follows inductively from

$$\widetilde{\phi}_{N_2}(t) \ge 0 = \overline{\phi}_{N_2}(t), \; \widetilde{\phi}_{N_2+1}(t) \ge 0 = \overline{\phi}_{N_2+1}(t)$$

and

$$\widetilde{\phi}_j(t) \geq \sup \left\{ \widetilde{c}_0 q^{-j}\sqrt{\ln n} + \min_{1\leq \ell \leq n} \widetilde{\phi}_{j+2}(t_\ell)\,; \begin{array}{l} n \in \mathbb{N}, t_1, \ldots, t_n \in B(t, q^{-j}), \\ d(t_\ell, t_k) > q^{-j-1}, \ell, k \leq n, \ell \neq k \end{array} \right\}.$$

In particular we get

$$\overline{\phi}_{N_1}^{N_2} \leq \frac{q^2}{q-8} c_0 \cdot \phi_{N_1}^{N_2} \leq \frac{q^2}{q-8} c_0 \cdot F,\ {}_{N_1}^{N_2+1}.$$

Using theorem 2.4.2 and proposition 2.1.2 we therefore obtain

$$\overline{\phi}_{N_1}^{N_2} \leq c\Theta_{N_1}^{N_2+1}.$$

Furthermore it follows from proposition 2.1.3 that

$$\Theta_{N_1}^{N_2+1} \geq q^{-N_1-1}\sqrt{\ln N(T, d, q^{-N_1-1})} \geq \frac{1}{q} \cdot q^{-N_1}\sqrt{\ln N(T, d, q^{-N_1})}.$$

Thus we get the desired result. □

2.5.4 Homogeneous Sets

We call T homogeneous with respect to (N_1, N_2, c_1) if for $N_1 \leq j \leq N_2 - 1$ and $t \in T$ the inequality

$$m_j^{c_1} \leq M(B(t, q^{-j}), d, q^{-j-1})$$

holds. Here we again denote by m_j the quantity $\sup\limits_{t\in T} M(B(t, q^{-j}), d, q^{-j-1})$. Note that for example every group with a suitable metric (see section 2.2.4) is homogeneous w.r.t. $(N_1, N_2, 1)$.

Proposition 2.5.4 If T is homogeneous w.r.t. (N_1, N_2, c_1), then

$$\frac{1}{\max\left\{q, \frac{2}{c_0\sqrt{c_1}}\right\}} \int_{q^{-N_2}}^{q^{-N_1+1}} \sqrt{\ln N(T, d, \varepsilon)}\, d\varepsilon \leq \overline{\phi}_{N_1}^{N_2} + q^{-N_1}\sqrt{\ln N(T, d, q^{-N_1})}.$$

Proof: We first show that

$$\overline{\phi}_j(t) \geq c_0\sqrt{c_1} \sum_{\substack{j \leq \ell \leq N_2-1 \\ \ell \equiv j \bmod 2}} q^{-\ell}\sqrt{\ln m_\ell}$$

for $N_1 \leq j \leq N_2 - 1$.

This can be seen inductively by using

$$\begin{aligned} \overline{\phi}_{N_2-1}(t) &= c_0 q^{-(N_2-1)}\sqrt{\ln M(B(t, q^{-(N_2-1)}), d, q^{-N_2})} \\ &\geq c_0\sqrt{c_1} q^{-(N_2-1)}\sqrt{\ln m_{N_2-1}}, \\ \overline{\phi}_{N_2-2}(t) &= c_0 q^{-(N_2-2)}\sqrt{\ln M(B(t, q^{-(N_2-2)}), d, q^{-(N_2-1)})} \\ &\geq c_0\sqrt{c_1} q^{-(N_2-2)}\sqrt{\ln m_{N_2-2}} \end{aligned}$$

and

$$\begin{aligned} \overline{\phi}_j(t) &\geq c_0 q^{-j}\sqrt{\ln M(B(t, q^{-j}), d, q^{-j-1})} + \min_{1 \leq \ell \leq n} \overline{\phi}_{j+2}(t_\ell) \\ &\geq c_0\sqrt{c_1} q^{-j}\sqrt{\ln m_j} + \min_{1 \leq \ell \leq n} \overline{\phi}_{j+2}(t_\ell) \end{aligned}$$

for $n = M(B(t, q^{-j}), d, q^{-j-1})$ and points $t_1, \ldots, t_n \in B(t, q^{-j})$ satisfying $d(t_\ell, t_k) > q^{-j-1}, 1 \leq \ell, k \leq n, \ell \neq k$.

Hence

$$\overline{\phi}_{N_1}^{N_2} \geq \frac{c_0\sqrt{c_1}}{2} \sum_{\ell=N_1}^{N_2-1} q^{-\ell}\sqrt{\ln m_\ell}.$$

Now we set again $n_j = \sup_{t \in T} N(B(t, q^{-j}), d, q^{-j-1})$ such that $n_j \leq m_j$ and

$$N(T, d, q^{-j}) \leq N(T, d, q^{-N_1}) \cdot n_{N_1} \cdot n_{N_1+1} \cdot \ldots \cdot n_{j-1}.$$

We obtain

$$\begin{aligned} &\int_{q^{-N_2}}^{q^{-N_1+1}} \sqrt{\ln N(t, d, \varepsilon)}\, d\varepsilon \\ \leq\ &(q-1) \sum_{j=N_1}^{N_2} q^{-j}\sqrt{\ln N(T, d, q^{-j})} \end{aligned}$$

$$
\begin{aligned}
&\le (q-1)\sum_{j=N_1}^{N_2} q^{-j}\left(\sqrt{\ln N(T,d,q^{-N_1})}+\sum_{\ell=N_1}^{j-1}\sqrt{\ln m_\ell}\right)\\
&\le q\cdot q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}+(q-1)\sum_{\ell=N_1}^{N_2-1}\sum_{j=\ell+1}^{N_2} q^{-j}\sqrt{\ln m_\ell}\\
&\le q\cdot q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}+\sum_{\ell=N_1}^{N_2-1} q^{-\ell}\sqrt{\ln m_\ell}\\
&\le q\cdot q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}+\frac{2}{c_0\sqrt{c_1}}\overline{\phi}_{N_1}^{N_2}
\end{aligned}
$$

as desired. $\square$

2.5.5 Example (Sequence)

Here we consider the example from section 2.1.4 again, but only for $q \ge \sqrt{2}$. Recall the definitions of T, d and σ_j. We choose k_j with $a_{k_j}^2 \le q^{-2j} - a_{\sigma_j+1}^2$ such that

$$
T = \mathbb{N} = \bigcup_{k=1}^{\sigma_j} B(k, q^{-j}) \cup B(k_j, q^{-j}),\ N_1 \le j \le N_2 - 1.
$$

Note that $B(k, q^{-j}) = \{k\}, k \le \sigma_j$ and $B(k, q^{-j}) \subseteq B(k_j, q^{-j}), k > \sigma_j$. Setting

$$
m_j = \sup_{k\in\mathbb{N}} M(B(k, q^{-j}), d, q^{-j-1})
$$

we obtain $m_j = M(B(k_j, q^{-j}), d, q^{-j-1})$.

Furthermore we get by the definition of $\overline{\phi}_j$ that

$$
\overline{\phi}_j(k) = \overline{\phi}_{j+2}(k) = \ldots = \left\{\begin{array}{cl} \overline{\phi}_{N_2}(k) & ;\quad j \equiv N_2 \bmod 2 \\ \overline{\phi}_{N_2+1}(k) & ;\quad j \equiv (N_2+1) \bmod 2 \end{array}\right\} = 0
$$

for $k \le \sigma_j$ and $\overline{\phi}_j(k) \le \overline{\phi}_j(k_j)$ for $k > \sigma_j$. Hence

$$
\sup\{\overline{\phi}_j(k); k \in \mathbb{N}\} = \overline{\phi}_j(k_j).
$$

Now we compute $\overline{\phi}_j(k_j)$:

$$\begin{aligned}
&\overline{\phi}_j(k_j)\\
= \;& \sup\left\{ c_0 q^{-j}\sqrt{\ln n} + \min_{1\le r\le n} \overline{\phi}_{j+2}(\ell_r)\,;\; \begin{array}{l} n\in\mathbb{N}, \ell_1,\dots,\ell_n \in B(k_j,q^{-j}),\\ d(\ell_r,\ell_s) > q^{-j-1}, r,s\le n, r\ne s \end{array} \right\}\\
= \;& \max\{K_1, K_2\},
\end{aligned}$$

where

$$K_1 = \sup\{\overline{\phi}_{j+2}(\ell);\, \ell\in B(k_j,q^{-j})\} = \overline{\phi}_{j+2}(k_{j+2})$$

and

$$K_2 = \sup\left\{ c_0 q^{-j}\sqrt{\ln n} + \min_{1\le r\le n} \overline{\phi}_{j+2}(\ell_r)\,;\; \begin{array}{l} n>1, \ell_1,\dots,\ell_n \in B(k_j,q^{-j}),\\ d(\ell_r,\ell_s) > q^{-j-1}, r,s\le n, r\ne s \end{array} \right\}$$

(of course K_2 does not exist for $m_j = 1$).

If $\ell, k \in B(k_{j+2}, q^{-j-2}), \ell\ne k$, then

$$d(\ell,k) = \sqrt{a_\ell^2 + a_k^2} \le \sqrt{a_{\sigma_{j+2}+1}^2 + a_{\sigma_{j+2}+2}^2} < \sqrt{2}q^{-j-2} \le q^{-j-1}$$

and hence $M(B(k_{j+2},q^{-j-2}), d, q^{-j-1}) = 1$.

Consequently, whenever $\ell_1,\dots,\ell_n \in B(k_j,q^{-j})$ with $d(\ell_r,\ell_s) > q^{-j-1}$ for $r,s\le n, r\ne s, n>1$, we find at least one number

$$\ell_r \in B(k_j,q^{-j}) \setminus B(k_{j+2},q^{-j-2}) = \{\sigma_j+1, \sigma_j+2,\dots,\sigma_{j+2}\}.$$

Using $\overline{\phi}_{j+2}(k) = 0$ for $k\le \sigma_{j+2}$ we obtain $\min_{1\le r\le n} \overline{\phi}_{j+2}(\ell_r) = 0$.

This yields

$$\begin{aligned}
K_2 \;&=\; \sup\left\{ c_0 q^{-j}\sqrt{\ln n}\,;\; \begin{array}{l} n>1, \ell_1,\dots,\ell_n \in B(k_j,q^{-j}),\\ d(\ell_r,\ell_s) > q^{-j-1}, r,s\le n, r\ne s \end{array} \right\}\\
&=\; c_0 q^{-j}\sqrt{\ln m_j}.
\end{aligned}$$

Hence

$$\begin{aligned}\overline{\phi}_j(k_j) &= \max\{c_0 q^{-j}\sqrt{\ln m_j},\, \overline{\phi}_{j+2}(k_{j+2})\}\\ &= \max\{c_0 q^{-\ell}\sqrt{\ln m_\ell}\,;\, j \le \ell < N_2, \ell \equiv j \bmod 2\}\end{aligned}$$

and

$$\overline{\phi}_{N_1}^{N_2} = \max\{c_0 q^{-\ell}\sqrt{\ln m_\ell}\,;\, N_1 \le \ell \le N_2 - 1\}.$$

2.6 Further Representations

2.6.1 Definition of $\mathcal{I}\Theta_{N_1}^{N_2}$

Set

$$\mathcal{I}\Theta_{N_1,\mu}^{N_2} = \mathcal{I}\Theta_{N_1,\mu}^{N_2}(T,d) = \sup_{t\in T} \int_{2q^{-N_2}}^{2q^{-N_1}} \sqrt{\ln \frac{1}{\mu\,(B(t,\varepsilon))}}\, d\varepsilon$$

for $\mu \in \mathcal{P}(T,d)$.

We define

$$\mathcal{I}\Theta_{N_1}^{N_2} = \mathcal{I}\Theta_{N_1}^{N_2}(T,d) = \inf\{\mathcal{I}\Theta_{N_1,\mu}^{N_2}\,;\, \mu \in \mathcal{P}(T,d)\}.$$

It is again sufficient to consider $N_1 \ge N_0$, because

$$\mathcal{I}\Theta_{N_1}^{N_2} = \mathcal{I}\Theta_{N_0}^{N_2}\,,\; N_1 < N_0 < N_2 \quad \text{and} \quad \mathcal{I}\Theta_{N_1}^{N_2} = 0\,,\; N_1 < N_2 \le N_0.$$

2.6.2 Sudakov-Dudley-Bounds and Relations between $\mathcal{I}\Theta_{N_1}^{N_2}$ and $\Theta_{N_1}^{N_2}$

Proposition 2.6.2 Let $q > 4$ and $N_1 \ge N_0$. Then

$$\left(\frac{1}{2} - \frac{2}{q}\right) \sup_{N_1 \le j \le N_2 - 1} q^{-j}\sqrt{\ln M(T,d,q^{-j})} \le \mathcal{I}\Theta_{N_1}^{N_2} \le c \int_{q^{-N_2-1}}^{q^{-N_1-1}} \sqrt{\ln N(T,d,\varepsilon)}\, d\varepsilon.$$

Theorem 2.6.2 Let $q > 8$ and $N_1 \geq N_0$. Then

$$c_1 \mathcal{I}\Theta_{N_1}^{N_2} \leq \Theta_{N_1}^{N_2} \leq c_2 \mathcal{I}\Theta_{N_1}^{N_2+1}.$$

Proofs: First we will prove the left hand side of theorem 2.6.2 for $q > 1$:

By proposition 2.1.2 it is sufficient to show that

$$\mathcal{I}\Theta_{N_1}^{N_2} \leq cM\Theta_{N_1}^{N_2}.$$

For that purpose let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\mu \in \mathcal{P}(T,d)$ be arbitrary.

If $A \in \mathcal{A}_j, N_1 \leq j \leq N_2$ then $N(A,d,q^{-j}) = 1$ and hence $D(A) \leq 2q^{-j}$.

Consequently $A_j(t) \subseteq B(t, 2q^{-j})$ holds for $t \in T$, $N_1 \leq j \leq N_2$. We obtain

$$\begin{aligned} M\Theta_{N_1,\mathcal{A},\mu}^{N_2} &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}} \\ &\geq \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu\left(B(t,2q^{-j})\right)}} \\ &\geq \sup_{t\in T} \sum_{j=N_1+1}^{N_2} \frac{1}{2(q-1)} \int\limits_{2q^{-j}}^{2q^{-j+1}} \sqrt{\ln \frac{1}{\mu\left(B(t,\varepsilon)\right)}}\, d\varepsilon \\ &= \frac{1}{2(q-1)} \mathcal{I}\Theta_{N_1,\mu}^{N_2}. \end{aligned}$$

Now we prove proposition 2.6.2.

For the left hand side let $\mu \in \mathcal{P}(T,d)$ be arbitrary. Fix j with $N_1 \leq j < N_2$ and choose points $s_1, \ldots, s_n$ with $n = M(T,d,q^{-j})$ and $d(s_\ell, s_k) > q^{-j}$ for $\ell, k \leq n, \ell \neq k$.

We are able to find some point s_ℓ with

$$\mu\left(B(s_\ell, \frac{q^{-j}}{2})\right) \leq \frac{1}{n},$$

because the balls $B(s_1, \frac{q^{-j}}{2}), \ldots, B(s_n, \frac{q^{-j}}{2})$ are disjoint. We obtain

$$\mathcal{I}\Theta_{N_1,\mu}^{N_2} \geq \int\limits_{2q^{-j-1}}^{\frac{1}{2}q^{-j}} \sqrt{\ln \frac{1}{\mu\left(B(s_\ell,\varepsilon)\right)}}\, d\varepsilon$$

$$\geq \left(\frac{1}{2}-\frac{2}{q}\right) q^{-j} \sqrt{\ln \frac{1}{\mu\left(B(s_\ell, \frac{q^{-j}}{2})\right)}}$$
$$\geq \left(\frac{1}{2}-\frac{2}{q}\right) q^{-j} \sqrt{\ln M(T,d,q^{-j})}.$$

The right hand side follows from $\mathcal{I}\Theta_{N_1}^{N_2} \leq c\Theta_{N_1}^{N_2}$ and proposition 2.1.3.

It remains to show the right hand side of theorem 2.6.2 for $q > 8$.

Let $\mu \in \mathcal{P}(T,d)$ be arbitrary. Set

$$\phi_j(t) = \sup \left\{ \int_{2q^{-N_2-1}}^{2q^{-j}} \sqrt{\ln \frac{1}{\mu\left(B(s,\varepsilon)\right)}}\, d\varepsilon \,;\, s \in T, d(s,t) \leq 2q^{-j} \right\}$$

for $N_1 \leq j \leq N_2+1$. Obviously,

$$\phi_{N_1}^{N_2} = \sup\{\phi_j(t)\,;\, t \in T, N_1 \leq j \leq N_2+1\} \leq \mathcal{I}\Theta_{N_1,\mu}^{N_2+1}.$$

We now prove that the functions $\phi_{N_1}, \ldots, \phi_{N_2+1}$ satisfy the assumption of theorem 2.3.2. Let $N_1 \leq j \leq N_2-1$ and take $t_1, \ldots, t_n \in B(t, q^{-j})$ with $d(t_\ell, t_k) > q^{-j-1}$, $\ell, k \leq n, \ell \neq k$.

According to the definition of $\phi_{j+2}(t_\ell)$ we consider $s_1, \ldots, s_n \in T$ with $d(s_\ell, t_\ell) \leq 2q^{-j-2}$, $1 \leq \ell \leq n$.

The balls $B\left(s_\ell, \frac{q-4}{2} q^{-j-2}\right)$, $1 \leq \ell \leq n$ are disjoint because

$$d(s_\ell, s_k) \geq d(t_\ell, t_k) - d(s_\ell, t_\ell) - d(s_k, t_k) > q^{-j-1} - 4q^{-j-2}.$$

Hence there exists a point s_ℓ with $\mu\left(B\left(s_\ell, \frac{q-4}{2} q^{-j-2}\right)\right) \leq \frac{1}{n}$.

Using

$$d(t, s_\ell) \leq d(t, t_\ell) + d(t_\ell, s_\ell) \leq q^{-j} + 2q^{-j-2} \leq 2q^{-j}$$

we obtain

$$\phi_j(t) \geq \int_{2q^{-N_2-1}}^{2q^{-j}} \sqrt{\ln \frac{1}{\mu\left(B(s_\ell,\varepsilon)\right)}}\, d\varepsilon$$

$$\geq \int\limits_{2q^{-N_2-1}}^{2q^{-j-2}} \sqrt{\ln \frac{1}{\mu\left(B(s_\ell,\varepsilon)\right)}}\, d\varepsilon + \int\limits_{2q^{-j-2}}^{\frac{q-4}{2}q^{-j-2}} \sqrt{\ln n}\, d\varepsilon.$$

Thus

$$\phi_j(t) \geq \frac{q-8}{2q^2} q^{-j}\sqrt{\ln n} + \min_{1\leq \ell\leq n} \phi_{j+2}(t_\ell)\,.$$

We can now apply theorem 2.3.2 and get

$$\Theta_{N_1}^{N_2} \leq c\left(\mathcal{I}\Theta_{N_1,\mu}^{N_2+1} + q^{-N_1} + q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}\right).$$

Furthermore proposition 2.6.2 yields

$$\mathcal{I}\Theta_{N_1,\mu}^{N_2+1} \geq \left(\frac{1}{2}-\frac{2}{q}\right) q^{-N_1}\sqrt{\ln N(T,d,q^{-N_1})}$$

and

$$\mathcal{I}\Theta_{N_1,\mu}^{N_2+1} \geq \left(\frac{1}{2}-\frac{2}{q}\right) q^{-N_1-1}\sqrt{\ln N(T,d,q^{-N_1-1})} \geq \left(\frac{1}{2}-\frac{2}{q}\right) q^{-N_1-1}\sqrt{\ln 2}.$$

Thus we get as desired $\Theta_{N_1}^{N_2} \leq c\mathcal{I}\Theta_{N_1}^{N_2+1}$. □

2.6.3 Definition of $\Theta_{N_1}^{N_2,\not\subseteq}$ and $M\Theta_{N_1}^{N_2,\not\subseteq}$

Recall the definitions of section 2.1.1. Set

$$\Theta_{N_1}^{N_2,\not\subseteq} = \Theta_{N_1}^{N_2,\not\subseteq}(T,d) = \inf\{\Theta_{N_1,\mathcal{A},\omega}^{N_2}\,;\, \mathcal{A}\in\mathcal{Z}_{N_1}^{N_2,\not\subseteq}, \omega\in\mathcal{G}(\mathcal{A})\}$$

and

$$M\Theta_{N_1}^{N_2,\not\subseteq} = M\Theta_{N_1}^{N_2,\not\subseteq}(T,d) = \inf\{M\Theta_{N_1,\mathcal{A},\mu}^{N_2}\,;\, \mathcal{A}\in\mathcal{Z}_{N_1}^{N_2,\not\subseteq}, \mu\in\mathcal{P}(T,d)\}.$$

It is again sufficient to consider $N_1 \geq N_0$, the arguments are the same as in section 2.1.1.

Proposition 2.6.3

(a) $\Theta_{N_1}^{N_2,\not\subseteq} \leq M\Theta_{N_1}^{N_2,\not\subseteq} \leq c\Theta_{N_1}^{N_2,\not\subseteq}$

(b) $\mathcal{I}\Theta_{N_1}^{N_2} \leq cM\Theta_{N_1}^{N_2,\not\subseteq}$

(c) $\Theta_{N_1}^{N_2,\not\subseteq} \leq \Theta_{N_1}^{N_2}$

Proof:

(a) Can be proved in exactly the same way as proposition 2.1.2.

(b) In the proof of the left hand side of theorem 2.6.2 we did not need that the sequence of partitions was increasing.

Hence the same proof works for $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2,\not\subseteq}$.

(c) This follows by definitions because $\mathcal{Z}_{N_1}^{N_2} \subseteq \mathcal{Z}_{N_1}^{N_2,\not\subseteq}$. □

2.6.4 Discretization of $\mathcal{I}\Theta_{N_1}^{N_2}$

We want to show that $\mathcal{I}\Theta_{N_1}^{N_2}$ and $\Theta_{N_1}^{N_2,\not\subseteq}$ are more closely related than $\mathcal{I}\Theta_{N_1}^{N_2}$ and $\Theta_{N_1}^{N_2}$. Namely, we will prove an inequality similar to the right hand side of theorem 2.6.2 without Talagrand's construction:

Theorem 2.6.4 Let $q > 7$ and $N_1 \geq N_0$, then

$$\Theta_{N_1}^{N_2,\not\subseteq} \leq c \cdot \mathcal{I}\Theta_{N_1+2}^{N_2+2}.$$

Proof: If $\mathcal{I}\Theta_{N_1+2}^{N_2+2} = \infty$ then there is nothing to prove. Thus we assume $\mathcal{I}\Theta_{N_1+2}^{N_2+2} < \infty$. Let $\mu \in \mathcal{P}(T,d)$ with $\mathcal{I}\Theta_{N_1+2,\mu}^{N_2+2} < \infty$. Put $c_1 = \frac{1}{3} - \frac{4}{3q}$ and $c_2 = \frac{1}{3} + \frac{2}{3q}$, thus $0 < c_1 < c_2$ and $c_1 + 2c_2 = 1$.

We will construct $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2,\not\subseteq}$ and $\omega \in \mathcal{G}(\mathcal{A})$ such that

$$\omega_j\left(A_j(t)\right) \geq \mu\left(B(t,(c_2-c_1)q^{-j})\right) \qquad \text{for } t \in T, N_1 \leq j \leq N_2,$$

i.e. we will obtain a connection between μ and ω which is more direct than any using Talagrand's construction.

Note first that we get by proposition 2.6.2 and our choice of $c_1 > \frac{1}{q}$ (here we need $q > 7$) that

$$M(T, d, c_1 q^{-j}) \leq M(T, d, q^{-N_2 - 1}) < \infty \quad \text{for } N_1 \leq j \leq N_2.$$

Fix j with $N_1 \leq j \leq N_2$.

We choose points $t_1, \ldots, t_m \in T$ with $m = M(T, d, c_1 q^{-j})$ and $d(t_\ell, t_k) > c_1 q^{-j}$, $\ell, k \leq m, \ell \neq k$. Thus (see section 1.2)

$$T \subseteq \bigcup_{\ell=1}^{m} B(t_\ell, c_1 q^{-j}).$$

We may numerate $\{t_1, \ldots, t_m\}$ such that

$$\mu\left(B(t_\ell, c_2 q^{-j})\right) \geq \mu\left(B(t_{\ell+1}, c_2 q^{-j})\right) \text{ for } 1 \leq \ell \leq m-1.$$

Set

$$B_k = B(t_k, c_1 q^{-j}) \setminus \bigcup_{\ell < k} B(t_\ell, c_1 q^{-j}), \quad 1 \leq k \leq m.$$

We will obtain $\mathcal{A}_j$, that combines these sets $B_1, \ldots, B_m$, with help of the following procedure (see section 4.2. in [Ad]):

Set $C_1 = B(t_1, c_2 q^{-j})$ and for $2 \leq k \leq m$

$$C_k = \begin{cases} \emptyset & ; \quad B(t_k, c_2 q^{-j}) \cap \bigcup_{\ell < k} C_\ell \neq \emptyset \\ B(t_k, c_2 q^{-j}) & ; \quad B(t_k, c_2 q^{-j}) \cap \bigcup_{\ell < k} C_\ell = \emptyset. \end{cases}$$

We now choose $\ell(k)$ for any $1 \leq k \leq m$ which is maximal under the condition

$$\ell(k) \leq k \text{ and } B(t_k, c_2 q^{-j}) \cap C_{\ell(k)} \neq \emptyset.$$

Hence $C_{\ell(k)} = B(t_{\ell(k)}, c_2 q^{-j})$.

We denote the set $\{\ell(1), \ldots, \ell(m)\}$ by L and set

$$\mathcal{A}_j = \left\{ \bigcup_{\ell(k)=\ell} B_k \,;\, \ell \in L \right\}.$$

Taking $s \in B(t_k, c_2 q^{-j}) \cap C_{\ell(k)}$ we obtain for $t \in B_k$

$$\begin{aligned} d(t, t_{\ell(k)}) &\leq d(t, t_k) + d(t_k, s) + d(s, t_{\ell(k)}) \\ &\leq c_1 q^{-j} + c_2 q^{-j} + c_2 q^{-j} \\ &= q^{-j}, \end{aligned}$$

thus

$$\bigcup_{\ell(k)=\ell} B_k \subseteq B(t_\ell, q^{-j})$$

for $\ell \in L$. That means $\mathcal{A}_j$ satisfies the conditions which we need to ensure that $\mathcal{A} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N_2}) \in \mathcal{Z}_{N_1}^{N_2, \not\subseteq}$.

Set

$$\omega_j \left(\bigcup_{\ell(k)=\ell} B_k \right) = \mu(C_\ell)$$

for $\ell \in L$.

Hence we get for $t \in B_k, 1 \leq k \leq m$

$$\begin{aligned} \omega_j\left(A_j(t)\right) &= \mu(C_{\ell(k)}) \\ &= \mu\left(B(t_{\ell(k)}, c_2 q^{-j})\right) \\ &\geq \mu\left(B(t_k, c_2 q^{-j})\right) \\ &\geq \mu\left(B(t, (c_2 - c_1) q^{-j})\right) \end{aligned}$$

as desired. Furthermore the sets $C_1, \ldots, C_m$ are disjoint and thus

$$\begin{aligned} \sum_{A \in \mathcal{A}_j} \omega_j(A) &= \sum_{\ell \in L} \omega_j \left(\bigcup_{\ell(k)=\ell} B_k \right) \\ &= \sum_{\ell \in L} \mu(C_\ell) \\ &\leq 1. \end{aligned}$$

It follows $\omega = (\omega_{N_1}, \ldots, \omega_{N_2}) \in \mathcal{G}(\mathcal{A})$.

Finally we obtain

$$\begin{aligned}
\Theta^{N_2}_{N_1,\mathcal{A},\omega} &= \sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{\omega_j(A_j(t))}} \\
&\le \sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{\mu\left(B(t,(c_2-c_1)q^{-j})\right)}} \\
&\le \sup_{t\in T}\sum_{j=N_1+1}^{N_2} \frac{1}{2(q^{-1}-q^{-2})}\int_{2q^{-j-2}}^{2q^{-j-1}} \sqrt{\ln\frac{1}{\mu\left(B(t,\varepsilon)\right)}}\,d\varepsilon \\
&= \frac{q^2}{2(q-1)}\mathcal{I}\Theta^{N_2+2}_{N_1+2,\mu}.
\end{aligned}$$

$\square$

2.6.5 Nonmeasurable Partitions

Recall the definitions of section 2.1.1. One may ask whether it is essential that the partitions of T were measurable. It is not as we will show now.

Denote by $N\mathcal{Z}^{N_2,\not\subseteq}_{N_1}$ ($N\mathcal{Z}^{N_2}_{N_1}$) the set of all sequences $\mathcal{A}=(\mathcal{A}_{N_1},\dots,\mathcal{A}_{N_2})$ satisfying

(a′) $\mathcal{A}_j$ is a finite partition of T, $N_1\le j\le N_2$

and **(b)** (**(b)** and **(c)**) from section 2.1.1.

We adopt the definitions of $\mathcal{G}(\mathcal{A})$ and $\Theta^{N_2}_{N_1,\mathcal{A},\omega}$ from section 2.1.1 and define

$$N\Theta^{N_2}_{N_1} = N\Theta^{N_2}_{N_1}(T,d) = \inf\{\Theta^{N_2}_{N_1,\mathcal{A},\omega}\,;\,\mathcal{A}\in N\mathcal{Z}^{N_2}_{N_1},\omega\in\mathcal{G}(\mathcal{A})\}$$

$$N\Theta^{N_2,\not\subseteq}_{N_1} = N\Theta^{N_2,\not\subseteq}_{N_1}(T,d) = \inf\{\Theta^{N_2}_{N_1,\mathcal{A},\omega}\,;\,\mathcal{A}\in N\mathcal{Z}^{N_2,\not\subseteq}_{N_1},\omega\in\mathcal{G}(\mathcal{A})\}.$$

Proposition 2.6.5

(a) $N\Theta^{N_2,\not\subseteq}_{N_1}\le N\Theta^{N_2}_{N_1}\le\Theta^{N_2}_{N_1}$

(b) $\mathcal{I}\Theta^{N_2}_{N_1}\le c\cdot N\Theta^{N_2,\not\subseteq}_{N_1}$.

Proof:

(a) This follows from the definitions because $\mathcal{Z}_{N_1}^{N_2} \subseteq N\mathcal{Z}_{N_1}^{N_2} \subseteq N\mathcal{Z}_{N_1}^{N_2,\not\subseteq}$.

(b) Let $\mathcal{A} \in N\mathcal{Z}_{N_1}^{N_2,\not\subseteq}$ and $\omega \in \mathcal{G}(\mathcal{A})$.

Recall the proof of the right hand side of proposition 2.1.2. The same construction which yields a suitable $\mu \in \mathcal{P}(T,d)$ using $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$ in 2.1.2 is applicable here if we use $\mathcal{A} \in N\mathcal{Z}_{N_1}^{N_2,\not\subseteq}$ and $\omega \in \mathcal{G}(\mathcal{A})$.

The important property of μ is that

$$\mu\left(B(t,2q^{-j})\right) \geq 2^{-j+N_1}\omega_j\left(A_j(t)\right).$$

Thus

$$\begin{aligned}
\mathcal{I}\Theta_{N_1,\mu}^{N_2} &= \sup_{t\in T}\sum_{j=N_1+1}^{N_2}\int_{2q^{-j}}^{2q^{-j+1}}\sqrt{\ln\frac{1}{\mu\left(B(t,\varepsilon)\right)}}\,d\varepsilon \\
&\leq 2(q-1)\sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{2^{-j+N_1}\omega_j\left(A_j(t)\right)}} \\
&\leq 2(q-1)\left(q\sum_{j=1}^{\infty} q^{-j}\sqrt{j}+1\right)\Theta_{N_1,\mathcal{A},\omega}^{N_2}.
\end{aligned}$$

$\square$

2.7 Ultrametric Spaces

2.7.1 Definition and Elementary Properties

A metric space (T,d') is said to be an ultrametric space if

$$\bigwedge_{r,s,t\in T}\left(d'(s,t) \leq \max\{d'(s,r),d'(r,t)\}\right).$$

We denote by $B'(t,\varepsilon)$ the ball w.r.t. d'.

Proposition 2.7.1

(a) $\bigwedge_{s,t\in T}\bigwedge_{\varepsilon>0}\big(B'(s,\varepsilon)=B'(t,\varepsilon)\ \vee\ B'(s,\varepsilon)\cap B'(t,\varepsilon)=\emptyset\big)$

(b) $\bigwedge_{S\subseteq T}\bigwedge_{\varepsilon>0}\big(N(S,d',\varepsilon)=M(S,d',\varepsilon)\big)$

Proof:

(a) For $B'(s,\varepsilon)\cap B'(t,\varepsilon)\neq\emptyset$ we choose $r\in B'(s,\varepsilon)\cap B'(t,\varepsilon)$. Then we obtain for any $u\in B'(s,\varepsilon)$

$$\begin{aligned} d'(u,t) &\le \max\{d'(u,r),d'(r,t)\} \\ &\le \max\{d'(u,s),d'(s,r),d'(r,t)\} \\ &\le \varepsilon. \end{aligned}$$

Hence $B'(s,\varepsilon)\subseteq B'(t,\varepsilon)$. Analogous we get $B'(t,\varepsilon)\subseteq B'(s,\varepsilon)$.
Thus $B'(s,\varepsilon)=B'(t,\varepsilon)$.

(b) It is sufficient to show $M(S,d',\varepsilon)\le N(S,d',\varepsilon)$.
Let $n=N(S,d',\varepsilon)$ and $S\subseteq\bigcup_{\ell=1}^{n}B'(t_\ell,\varepsilon)$ for suitable points $t_1,\dots,t_n\in T$.
If $1\le\ell\le n$ and $s,t\in S\cap B'(t_\ell,\varepsilon)$ then

$$d'(s,t)\le\max\{d'(s,t_\ell),d'(t_\ell,t)\}\le\max\{\varepsilon,\varepsilon\}=\varepsilon,$$

hence $M(S,d',\varepsilon)\le n$. □

2.7.2 $U\Theta_{N_1}^{N_2}$

We define

$$U\Theta_{N_1}^{N_2}=U\Theta_{N_1}^{N_2}(T,d)=\inf\{\mathcal{I}\Theta_{N_1}^{N_2}(T,d')\,;\,d'\text{ ultrametric },\,d'\ge d\}.$$

There always exists an ultrametric $d'\ge d$, namely

$$d'(s,t)=\begin{cases}0 & ;\quad s=t\\ 2q^{-N_0} & ;\quad s\neq t.\end{cases}$$

It is again sufficient to consider $N_1 \geq N_0$:

Let $N_1 < N_0$ and choose an arbitrary ultrametric d' on T with $d' \geq d$. Setting

$$d''(s,t) = \min\{d'(s,t), 2q^{-N_0}\}\,,\; s,t \in T,$$

we obtain an ultrametric d'' on T with $d'' \geq d$ and

$$B''(t,\varepsilon) = \begin{cases} B'(t,\varepsilon) & ; \quad \varepsilon < 2q^{-N_0} \\ T & ; \quad \varepsilon \geq 2q^{-N_0}. \end{cases}$$

We denote by $B'(t,\varepsilon)$ and $B''(t,\varepsilon)$ the balls w.r.t. d' and d'', respectively. If $\mu \in \mathcal{P}(T,d')$, then $\mu \in \mathcal{P}(T,d'')$ and

$$\begin{aligned} \mathcal{I}\Theta^{N_2}_{N_1,\mu}(T,d'') &= \sup_{t\in T} \int_{2q^{-N_2}}^{2q^{-N_1}} \sqrt{\ln \frac{1}{\mu\left(B''(t,\varepsilon)\right)}}\, d\varepsilon \\ &= \begin{cases} \mathcal{I}\Theta^{N_2}_{N_0,\mu}(T,d') & ; \quad N_2 > N_0 \\ 0 & ; \quad N_2 \leq N_0. \end{cases} \end{aligned}$$

Thus

$$\mathcal{I}\Theta^{N_2}_{N_1}(T,d'') \leq \begin{cases} \mathcal{I}\Theta^{N_2}_{N_0}(T,d') & ; \quad N_2 > N_0 \\ 0 & ; \quad N_2 \leq N_0. \end{cases}$$

Hence we obtain $U\Theta^{N_2}_{N_1}(T,d) = 0$ for $N_2 \leq N_0$ and

$$U\Theta^{N_2}_{N_0}(T,d) \leq U\Theta^{N_2}_{N_1}(T,d) \leq \mathcal{I}\Theta^{N_2}_{N_1}(T,d'') \leq \mathcal{I}\Theta^{N_2}_{N_0}(T,d').$$

That means $U\Theta^{N_2}_{N_1}(T,d) = U\Theta^{N_2}_{N_0}(T,d)$ for $N_2 > N_0$.

Theorem 2.7.2 Let $N_1 \geq N_0$ and $q > 4$, then

$$\frac{2(q-1)}{q^2} \cdot N\Theta^{N_2-2}_{N_1-2} \leq U\Theta^{N_2}_{N_1} \leq 2(q-1) \cdot M\Theta^{N_2}_{N_1}.$$

Proof: Let us first prove the left hand side. For $U\Theta_{N_1}^{N_2} = \infty$ there is nothing to show. From now on we assume $U\Theta_{N_1}^{N_2} < \infty$, and we consider an ultrametric $d' \geq d$ and $\mu \in \mathcal{P}(T, d')$ with $\mathcal{I}\Theta_{N_1,\mu}^{N_2}(T, d') < \infty$.
We fix some points t_ℓ^j, $1 \leq \ell \leq N(T, d', q^{-j})$, $N_1 - 2 \leq j \leq N_2 - 2$ with

$$T \subseteq \bigcup_\ell B'(t_\ell^j, q^{-j}).$$

Set

$$\mathcal{A}_j = \left\{B'(t_\ell^j, q^{-j})\,;\, \ell = 1, \ldots, N(T, d', q^{-j})\right\},\ N_1 - 2 \leq j \leq N_2 - 2.$$

$\mathcal{A}_j$ is a finite partition of T because d' is an ultrametric (see proposition 2.7.1(a)) and

$$N(T, d', q^{-j}) \leq N(T, d', q^{-(N_2-1)}) < \infty$$

by proposition 2.6.2.
Furthermore $B'(t_\ell^j, q^{-j}) \subseteq B(t_\ell^j, q^{-j})$ (hence $N(B'(t_\ell^j, q^{-j}), d, q^{-j}) = 1$) and

$$A_{j+1}(t) \subseteq A_j(t)\,,\ N_1 - 2 \leq j < N_2 - 2, t \in T.$$

The latter follows from $A_j(t) = B'(t, q^{-j})$ (use again proposition 2.7.1(a)).
Thus $\mathcal{A} = (\mathcal{A}_{N_1-2}, \ldots, \mathcal{A}_{N_2-2}) \in N\mathcal{Z}_{N_1-2}^{N_2-2}$.
Set

$$\omega_j\left(B'(t_\ell^j, q^{-j})\right) = \mu\left(B'(t_\ell^j, q^{-j})\right)\,,\ \ell \leq N(T, d', q^{-j})$$

for $N_1 - 2 \leq j \leq N_2 - 2$. This yields $\omega = (\omega_{N_1-2}, \ldots, \omega_{N_2-2}) \in \mathcal{G}(\mathcal{A})$.
We obtain

$$\begin{aligned}
\mathcal{I}\Theta_{N_1,\mu}^{N_2}(T, d') &\geq 2(q-1)\sup_{t\in T}\sum_{j=N_1+1}^{N_2} q^{-j}\sqrt{\ln\frac{1}{\mu\left(B'(t, 2q^{-j+1})\right)}} \\
&\geq \frac{2(q-1)}{q^2}\sup_{t\in T}\sum_{j=N_1-1}^{N_2-2} q^{-j}\sqrt{\ln\frac{1}{\mu\left(B'(t, q^{-j})\right)}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(q-1)}{q^2} \sup_{t\in T} \sum_{j=N_1-1}^{N_2-2} q^{-j} \sqrt{\ln \frac{1}{\omega_j\left(A_j(t)\right)}} \\
&= \frac{2(q-1)}{q^2} \cdot \Theta_{N_1-2,\mathcal{A},\omega}^{N_2-2}.
\end{aligned}
$$

For the right hand side let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\mu \in \mathcal{P}(T,d)$ be arbitrary.

Set

$$
d'(s,t) = \begin{cases} 0 & ; \quad s = t \\ 2q^{-N_2} & ; \quad A_{N_2}(s) = A_{N_2}(t) \wedge s \neq t \\ 2q^{-j} & ; \quad A_j(s) = A_j(t) \wedge A_{j+1}(s) \neq A_{j+1}(t) \quad (N_1 \leq j < N_2) \\ 2q^{-N_0} & ; \quad A_{N_1}(s) \neq A_{N_1}(t) \end{cases}
$$

for $s,t \in T$. Obviously,

$$d'(s,t) \geq d(s,t),$$

$$d'(s,t) \geq 0,\ d'(s,t) = 0 \iff s = t \text{ and}$$

$$d'(s,t) = d'(t,s).$$

To prove that d' is an ultrametric with $d' \geq d$, it remains to show

$$d'(s,t) \leq \max\{d'(s,r), d'(r,t)\} \text{ for } r,s,t \in T.$$

The case $0 \in \{d'(s,t), d'(s,r), d'(r,t)\}$ is trivial. Thus we assume

$$d'(s,t) = 2q^{-j},\ d'(s,r) = 2q^{-\ell} \text{ and } d'(r,t) = 2q^{-k}$$

with $j,\ell,k \in \{N_0, N_1, N_1+1, \ldots, N_2\}$.

If $j = N_0$ then $A_{N_1}(s) \neq A_{N_1}(t)$ and hence

$$A_{N_1}(s) \neq A_{N_1}(r) \text{ or } A_{N_1}(r) \neq A_{N_1}(t),$$

i.e. $\ell = N_0$ or $k = N_0$.

Let now $N_1 \leq j \leq N_2$. We may suppose that $\ell \leq k$. If $\ell = N_0$ we are finished. If $N_0 < \ell$ then $A_\ell(s) = A_\ell(r)$ and $A_k(r) = A_k(t)$, hence (increasing sequence) $A_\ell(r) = A_\ell(t)$. Thus $A_\ell(s) = A_\ell(t)$ and $d'(s,t) \leq 2q^{-\ell}$.

Note that

$$B'(t,\varepsilon) = \begin{cases} \{t\} & ; \quad 0 \leq \varepsilon < 2q^{-N_2} \\ A_j(t) & ; \quad 2q^{-j} \leq \varepsilon < 2q^{-j+1} \quad (N_1 < j \leq N_2) \\ A_{N_1}(t) & ; \quad 2q^{-N_1} \leq \varepsilon < 2q^{-N_0} \\ T & ; \quad 2q^{-N_0} \leq \varepsilon. \end{cases}$$

Hence $\mu \in \mathcal{P}(T,d')$ and we obtain

$$\begin{aligned} \mathcal{I}\Theta_{N_1,\mu}^{N_2}(T,d') &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} \int_{2q^{-j}}^{2q^{-j+1}} \sqrt{\ln \frac{1}{\mu\left(B'(t,\varepsilon)\right)}}\, d\varepsilon \\ &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} \int_{2q^{-j}}^{2q^{-j+1}} \sqrt{\ln \frac{1}{\mu\left(A_j(t)\right)}}\, d\varepsilon \\ &= 2(q-1) \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu\left(A_j(t)\right)}} \\ &= 2(q-1) \cdot M\Theta_{N_1,\mathcal{A},\mu}^{N_2}. \end{aligned}$$

□

2.7.3 $\overline{\phi}_{N_1}^{N_2}$ and Trees

Let $N_2 < \infty$ and (T,d') be an ultrametric space.

We construct a tree (V,E) :

Fix points $t_\ell^j \in T$, $1 \leq \ell \leq N(T,d',q^{-j})$, $N_1 \leq j \leq N_2$, with

$$T \subseteq \bigcup_\ell B'(t_\ell^j, q^{-j}).$$

Denote $B'(t_\ell^j, q^{-j})$ by B_ℓ^j. Note that the set $\{B_\ell^j\,;\, 1 \leq \ell \leq N(T,d',q^{-j})\}$ is uniquely determined (see proposition 2.7.1(a)).

Set

$$V = \{T, B^j_\ell\,;\, 1 \le \ell \le N(T, d', q^{-j}), N_1 \le j \le N_2\}$$

and denote by E (edges) the union of the following two sets:

$$\{(T, B^{N_1}_\ell)\,;\, 1 \le \ell \le N(T, d', q^{-N_1})\}$$

and

$$\left\{(B^j_\ell, B^{j+1}_k)\,;\, \begin{array}{l} N_1 \le j < N_2, B^{j+1}_k \subseteq B^j_\ell, \\ \ell \le N(T, d', q^{-j}), k \le N(T, d', q^{-j-1}) \end{array}\right\}.$$

Recall the definition of $\overline{\phi}^{N_2}_{N_1}(T, d')$. Let $N_1 \le j \le N_2 - 1$.

If $B'(t, q^{-j}) = B'(s, q^{-j})$ then $\overline{\phi}_j(t) = \overline{\phi}_j(s)$, thus $\overline{\phi}_j(t) = \overline{\phi}_j(t^j_\ell)$ for $t \in B^j_\ell$.

Set

$$\begin{aligned} \overline{\phi}_{j+2}(B^{j+1}_\ell) &= \sup\{\overline{\phi}_{j+2}(t)\,;\, t \in B^{j+1}_\ell\} \\ &= \begin{cases} \sup\{\overline{\phi}_{j+2}(t^{j+2}_k)\,;\, (B^{j+1}_\ell, B^{j+2}_k) \in E\} & ; \quad j < N_2 - 1 \\ 0 & ; \quad j = N_2 - 1. \end{cases} \end{aligned}$$

To compute $\overline{\phi}_j(t)$ let us denote the set B^j_ℓ, which contains t, by C and by

$$\{C_1, \ldots, C_m\}\,,\ m = N(C, d', q^{-j-1}) = M(C, d', q^{-j-1})$$

the set

$$\{B^{j+1}_k\,;\, (B^j_\ell, B^{j+1}_k) \in E\}.$$

Note that for $r, s \in C$ we get

$$d'(r, s) > q^{-j-1} \iff \bigvee_{\ell,k} \big(\ell, k \le m, \ell \ne k, r \in C_\ell, s \in C_k\big).$$

Number the set $\{C_1, \ldots, C_m\}$ such that

$$\overline{\phi}_{j+2}(C_1) \ge \overline{\phi}_{j+2}(C_2) \ge \ldots \ge \overline{\phi}_{j+2}(C_m).$$

We obtain

$$\overline{\phi}_j(t) = \sup\{c_0 q^{-j}\sqrt{\ln n} + \overline{\phi}_{j+2}(C_n)\,;\, 1 \le n \le m\}.$$

2.8 Shifting of Indices

In this section we investigate the shifting of indices more closely because we have seen the importance of it.

2.8.1 Lower Index

Proposition 2.8.1 For $N_2 \geq N_1 + 2$ it holds

$$M\Theta_{N_1+1}^{N_2} \leq M\Theta_{N_1}^{N_2} \leq (1+q) \cdot M\Theta_{N_1+1}^{N_2}.$$

Proof: Let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\mu \in \mathcal{P}(T,d)$ be arbitrary.

Set $\widetilde{\mathcal{A}} = (\mathcal{A}_{N_1+1}, \ldots, \mathcal{A}_{N_2})$. Obviously, $M\Theta_{N_1+1,\widetilde{\mathcal{A}},\mu}^{N_2} \leq M\Theta_{N_1,\mathcal{A},\mu}^{N_2}$. This yields the left hand side.

For the right hand side let $\mathcal{A} \in \mathcal{Z}_{N_1+1}^{N_2}$ and $\mu \in \mathcal{P}(T,d)$ be arbitrary.

Set $\widetilde{\mathcal{A}} = (\mathcal{A}_{N_1+1}, \mathcal{A}_{N_1+1}, \ldots, \mathcal{A}_{N_2})$. We use $A_{N_1+2}(t) \subseteq A_{N_1+1}(t)$ and hence $\mu\left(A_{N_1+2}(t)\right) \leq \mu\left(A_{N_1+1}(t)\right)$. Thus we obtain

$$\begin{aligned}
M\Theta_{N_1,\widetilde{\mathcal{A}},\mu}^{N_2} &= \sup_{t\in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu(A_j(t))}} \\
&\leq \sup_{t\in T} \left(\sum_{j=N_1+2}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\mu\left(A_j(t)\right)}} + q^{-N_1-1} \sqrt{\ln \frac{1}{\mu\left(A_{N_1+2}(t)\right)}} \right) \\
&\leq (1+q) \cdot M\Theta_{N_1+1,\mathcal{A},\mu}^{N_2}.
\end{aligned}$$

□

2.8.2 Upper Index

Proposition 2.8.2

$$\Theta_{N_1}^{N_2} \leq \Theta_{N_1}^{N_2+1} \leq \left(1 + \frac{1}{q}\right) \Theta_{N_1}^{N_2} + q^{-(N_2+1)} \sqrt{\ln N(T,d,q^{-(N_2+1)})}$$

Proof: For the left hand side let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2+1}$ and $\omega \in \mathcal{G}(\mathcal{A})$ be arbitrary. Set $\widetilde{\mathcal{A}} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N_2})$ and $\widetilde{\omega} = (\omega_{N_1}, \ldots, \omega_{N_2})$.
Thus we get $\Theta_{N_1,\widetilde{\mathcal{A}},\widetilde{\omega}}^{N_2} \leq \Theta_{N_1,\mathcal{A},\omega}^{N_2+1}$.
For the right hand side let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$ be arbitrary. Choose a partition $\mathcal{B}$ of T which is induced by a covering with $N(T, d, q^{-(N_2+1)})$ balls of radius $q^{-(N_2+1)}$. Set

$$\mathcal{A}_{N_2+1} = \{A \cap B\,;\; A \cap B \neq \emptyset, A \in \mathcal{A}_{N_2}, B \in \mathcal{B}\}$$

and

$$\omega_{N_2+1}(A \cap B) = \frac{\omega_{N_2}(A)}{N(T, d, q^{-(N_2+1)})}.$$

Of course,

$$\widetilde{\mathcal{A}} = (\mathcal{A}_{N_1}, \ldots, \mathcal{A}_{N_2}, \mathcal{A}_{N_2+1}) \in \mathcal{Z}_{N_1}^{N_2+1} \text{ and } \widetilde{\omega} = (\omega_{N_1}, \ldots, \omega_{N_2}, \omega_{N_2+1}) \in \mathcal{G}(\widetilde{\mathcal{A}}).$$

Furthermore we obtain

$$\begin{aligned}
\Theta_{N_1,\widetilde{\mathcal{A}},\widetilde{\omega}}^{N_2+1} &= \sup_{t \in T} \left(\sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j\,(A_j(t))}} + q^{-N_2-1} \sqrt{\ln \frac{N(T, d, q^{-(N_2+1)})}{\omega_{N_2}\,(A_{N_2}(t))}} \right) \\
&\leq \left(1 + \frac{1}{q}\right) \Theta_{N_1,\mathcal{A},\omega}^{N_2} + q^{-(N_2+1)} \sqrt{\ln N(T, d, q^{-(N_2+1)})}
\end{aligned}$$

as desired. □

2.9 Generalization

We want to remark that most proofs of this chapter also work with minor changes if

$$\sqrt{\ln \cdot} \qquad \text{is replaced by} \qquad f(\cdot),$$

where f is a function satisfying the assumption of lemma 2.4.2. Therefore we included this lemma in the stated form although we used it only for $f(x) = \sqrt{\ln x}$.

Moreover, it is possible to replace

$$q^{-j}\sqrt{\ln\frac{1}{\omega_j\left(A_j(t)\right)}} \qquad \text{by} \qquad q^{-\beta j} f\left(\frac{1}{\omega_j\left(A_j(t)\right)}\right),$$

$$\sqrt{\ln\frac{1}{\mu\left(B(t,\varepsilon)\right)}} \qquad \text{by} \qquad \varepsilon^{\beta-1} f\left(\frac{1}{\mu\left(B(t,\varepsilon)\right)}\right)$$

and so on for some fixed $\beta > 0$. Note that we then have to replace in Talagrand's construction

$$c_0 q^{-j}\sqrt{\ln n} \qquad \text{by} \qquad c_0 q^{-\beta j} f(n).$$

Chapter 3

Applications to Gaussian Random Functions

Recall the situation of section 1.1, in particular we assume now that $T \subseteq \ell_2$ and $d(s,t) = \|s-t\|_2$.

3.1 Background Theorems

Theorem 3.1.1

$$\mathbb{E} \sup_{\substack{s,t \in T \\ d(s,t) \leq 2q^{-N_1}}} |X_s - X_t| \leq c \inf_{\mu \in \mathcal{P}(T,d)} \sup_{t \in T} \int_0^{2q^{-N_1}} \sqrt{\ln \frac{1}{\mu\left(B(t,\varepsilon)\right)}}\, d\varepsilon$$

Proof: This theorem is a direct consequence of [F2], 5.2.6. □

Theorem 3.1.2 If $t_1, \ldots, t_n \in T$ with $d(t_\ell, t_k) > 4\varepsilon$, $\ell, k \leq n$, $\ell \neq k$, then

$$\mathbb{E} \sup_{t \in T} X_t \geq \frac{2}{3}\varepsilon\sqrt{\ln n} + \min_{1 \leq \ell \leq n} \mathbb{E} \sup_{t \in B(t_\ell, \varepsilon)} X_t.$$

Proof: See [F2], 3.4.5. □

Theorem 3.1.3 Let $\{t_1, \ldots, t_n\}$ and $\{s_1, \ldots, s_n\}$ be subsets of ℓ_2 with

$$d(s_i, s_j) \leq d(t_i, t_j),\ 1 \leq i, j \leq n, i \neq j.$$

Then it holds

$$\mathbb{E} \sup_{1\leq i\leq n} X_{s_i} \leq \mathbb{E} \sup_{1\leq i\leq n} X_{t_i}$$

and

$$\mathbb{E} \sup_{1\leq i,j\leq n} |X_{s_i} - X_{s_j}| \leq \mathbb{E} \sup_{1\leq i,j\leq n} |X_{t_i} - X_{t_j}|.$$

Proof: See [F2], 3.2.5, or [Ka]. □

Theorem 3.1.4 If $(s_n)_{n=1}^{\infty}$ is a sequence in ℓ_2 satisfying $\|s_n\|_2 \leq \frac{1}{\sqrt{\ln(n+1)}}$, then $\mathbb{E} \sup_n |X_{s_n}| \leq c$.

Proof: See section 8 in [T2]. □

3.2 Relations between $CF_{N_1}^{N_2}$ and $\Theta_{N_1}^{N_2}$

Set $\mathcal{C}_{N_2} = \left\{ T_{N_2} \subseteq T\,;\, T \subseteq \bigcup_{r\in T_{N_2}} B(r, 2q^{-N_2}) \right\}$. Note that $\mathcal{C}_\infty = \{T\}$.
Define

$$CF_{N_1}^{N_2}(T) = \inf \left\{ \sup_{t\in T_{N_2}} \mathbb{E} \sup_{\substack{s\in T_{N_2} \\ d(s,t)\leq 2q^{-N_1}}} X_s\,;\ T_{N_2} \in \mathcal{C}_{N_2} \right\}$$

and

$$\overline{CF}_{N_1}^{N_2}(T) = \inf \left\{ \mathbb{E} \sup_{\substack{s,t\in T_{N_2} \\ d(s,t)\leq 2q^{-N_1}}} |X_s - X_t|\,;\ T_{N_2} \in \mathcal{C}_{N_2} \right\}.$$

It is sufficient to consider $N_1 \geq N_0$ because $D(T) \leq 2q^{-N_0}$.

Proposition 3.2 Let $N_0 \le N_1 < N_2$ and $q > 28$ then

$$\begin{aligned} c_1 \Theta_{N_1+1}^{N_2-1}(T,d) &\le CF_{N_1}^{N_2}(T) + q^{-N_1-1}\sqrt{\ln N(T,d,q^{-N_1-1})} \\ &\le \overline{CF}_{N_1}^{N_2}(T) + q^{-N_1-1}\sqrt{\ln N(T,d,q^{-N_1-1})} \\ &\le c_2 \Theta_{N_1}^{N_2}(T,d). \end{aligned}$$

Proof: For $t \in T_{N_2}$, we get

$$\mathbb{E} \sup_{\substack{s\in T_{N_2} \\ d(s,t)\le 2q^{-N_1}}} X_s = \mathbb{E} \sup_{\substack{s\in T_{N_2} \\ d(s,t)\le 2q^{-N_1}}} (X_s - X_t) \le \mathbb{E} \sup_{\substack{s,t\in T_{N_2} \\ d(s,t)\le 2q^{-N_1}}} |X_s - X_t|$$

and thus $CF_{N_1}^{N_2}(T) \le \overline{CF}_{N_1}^{N_2}(T)$.

We now prove the right hand side.

By proposition 2.1.3 it is sufficient to show that $\overline{CF}_{N_1}^{N_2}(T) \le c\Theta_{N_1}^{N_2}(T,d)$. For that purpose let $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}$ and $\omega \in \mathcal{G}(\mathcal{A})$ be arbitrary. For any $A \in \mathcal{A}_{N_2}$ we choose an element $t_A \in A$ and denote the set $\{t_A\,;\, A \in \mathcal{A}_{N_2}\}$ by T_{N_2}. Obviously, $T_{N_2} \in \mathcal{C}_{N_2}$.

We define $\widetilde{\mathcal{A}} \in \mathcal{Z}_{N_1}^{\infty}(T_{N_2}, d_{|T_{N_2}\times T_{N_2}})$ and $\widetilde{\omega} \in \mathcal{G}(\widetilde{\mathcal{A}})$ as follows:

Set

$$\widetilde{\mathcal{A}}_j = \begin{cases} \{A \cap T_{N_2}\,;\, A \in \mathcal{A}_{j+1}\} & ; \quad N_1 \le j < N_2 \\ \{\{t\}\,;\, t \in T_{N_2}\} & ; \quad N_2 \le j < \infty. \end{cases}$$

For $j < N_2$ we need $A \in \mathcal{A}_{j+1}$ to ensure that

$$A \cap T_{N_2} \subseteq B(t, q^{-j})$$

for a suitable $t \in T_{N_2}$:

of course, $A \cap T_{N_2} \subseteq B(s, q^{-j-1})$ for a point $s \in T$. Now we can find $t \in T_{N_2}$ with $d(s,t) \le 2q^{-N_2}$. Using $q \ge 3$ we get

$$A \cap T_{N_2} \subseteq B(t, q^{-j-1} + 2q^{-N_2}) \subseteq B(t, q^{-j}).$$

Thus $\widetilde{\mathcal{A}} = (\widetilde{\mathcal{A}}_{N_1}, \widetilde{\mathcal{A}}_{N_1+1}, \ldots) \in \mathcal{Z}_{N_1}^{\infty}(T_{N_2}, d_{|T_{N_2}\times T_{N_2}})$.

Set

$$\widetilde{\omega}_j(A \cap T_{N_2}) = \omega_{j+1}(A) \text{ for } N_1 \leq j < N_2, A \in \mathcal{A}_{j+1}$$

and

$$\widetilde{\omega}(\{t\}) = \omega_{N_2}(A) \text{ for } j \geq N_2 \text{ and } t = t_A \in T_{N_2}.$$

Thus $\widetilde{\omega} = (\widetilde{\omega}_{N_1}, \widetilde{\omega}_{N_1+1}, \ldots) \in \mathcal{G}(\widetilde{\mathcal{A}})$.

By use of theorem 3.1.1 and theorem 2.6.2 we obtain

$$\begin{aligned}
& \overline{CF}_{N_1}^{N_2}(T) \\
\leq \; & \mathbb{E} \sup_{\substack{s,t \in T_{N_2} \\ d(s,t) \leq 2q^{-N_1}}} |X_s - X_t| \\
\leq \; & c \cdot \mathcal{I}\Theta_{N_1}^{\infty}(T_{N_2}, d_{|T_{N_2} \times T_{N_2}}) \\
\leq \; & c \cdot \Theta_{N_1}^{\infty}(T_{N_2}, d_{|T_{N_2} \times T_{N_2}}) \\
\leq \; & c \cdot \Theta_{N_1, \widetilde{\mathcal{A}}, \widetilde{\omega}}^{\infty}(T_{N_2}, d_{|T_{N_2} \times T_{N_2}}) \\
= \; & c \sup_{t \in T_{N_2}} \left(\sum_{j=N_1+1}^{N_2-1} q^{-j} \sqrt{\ln \frac{1}{\omega_{j+1}\left(A_{j+1}(t)\right)}} + \sum_{j=N_2}^{\infty} q^{-j} \sqrt{\ln \frac{1}{\omega_{N_2}\left(A_{N_2}(t)\right)}} \right) \\
= \; & c \sup_{t \in T_{N_2}} \left(q \sum_{j=N_1+2}^{N_2} q^{-j} \sqrt{\ln \frac{1}{\omega_j\left(A_j(t)\right)}} + \frac{q}{q-1} q^{-N_2} \sqrt{\ln \frac{1}{\omega_{N_2}\left(A_{N_2}(t)\right)}} \right) \\
\leq \; & c \cdot \Theta_{N_1, \mathcal{A}, \omega}^{N_2}(T, d).
\end{aligned}$$

It remains to prove the left hand side provided that $N_1 + 1 < N_2 - 1$.

Set

$$\mathcal{C}_{N_2}(t,j) = \left\{ R_{N_2} \subseteq T\, ;\; B(t, 2q^{-j}) \subseteq \bigcup_{r \in R_{N_2}} B(r, 2q^{-N_2}) \right\}.$$

We define for $N_1 + 1 \leq j \leq N_2$

$$\phi_j(t) = \inf \left\{ \mathbb{E} \sup_{r \in R_{N_2}} X_r\, ;\; R_{N_2} \in \mathcal{C}_{N_2}(t,j) \right\}.$$

We first prove that

$$\phi_{N_1+1}^{N_2-1} = \sup\{\phi_j(t)\, ;\, t \in T, N_1 + 1 \leq j \leq N_2\} \leq CF_{N_1}^{N_2}(T) :$$

Fix $T_{N_2} \in \mathcal{C}_{N_2}$. We have to show that

$$\phi_j(t) \leq \sup_{s \in T_{N_2}} \mathbb{E} \sup_{\substack{r \in T_{N_2} \\ d(r,s) \leq 2q^{-N_1}}} X_r \ , \ t \in T, N_1 + 1 \leq j \leq N_2.$$

For that purpose set

$$R_{N_2} = \left\{ r \in T_{N_2} \, ; \, B(t, 2q^{-j}) \cap B(r, 2q^{-N_2}) \neq \emptyset \right\}.$$

Obviously, $R_{N_2} \in \mathcal{C}_{N_2}(t,j)$. Fix $s \in R_{N_2}$.

Now we get for $r \in R_{N_2}$

$$d(r,s) \leq d(r,t) + d(t,s) \leq 2q^{-j} + 2q^{-N_2} + 2q^{-j} + 2q^{-N_2} \leq 8q^{-N_1-1} \leq 2q^{-N_1}$$

and hence

$$\phi_j(t) \leq \mathbb{E} \sup_{r \in R_{N_2}} X_r \leq \mathbb{E} \sup_{\substack{r \in T_{N_2} \\ d(r,s) \leq 2q^{-N_1}}} X_r \leq \sup_{s \in T_{N_2}} \mathbb{E} \sup_{\substack{r \in T_{N_2} \\ d(r,s) \leq 2q^{-N_1}}} X_r$$

as desired. Our final aim is to apply theorem 2.3.2. Thus we have to show that for $N_1 + 1 \leq j \leq N_2 - 2$ and any points $t_1, \ldots, t_n \in B(t, q^{-j})$ satisfying $d(t_\ell, t_k) > q^{-j-1}, \ell, k \leq n, \ell \neq k$ the inequality

$$\phi_j(t) \geq c_0 q^{-j} \sqrt{\ln n} + \min_{1 \leq \ell \leq n} \phi_{j+2}(t_\ell)$$

holds. We will prove it for $c_0 = \frac{4}{q^2}$. For that purpose let $R_{N_2} \in \mathcal{C}_{N_2}(t,j)$ be arbitrary. Choose $s_1, \ldots, s_n \in R_{N_2}$ with $d(t_\ell, s_\ell) \leq 2q^{-N_2}$, $1 \leq \ell \leq n$. This is always possible because

$$t_\ell \in B(t, q^{-j}) \subseteq B(t, 2q^{-j}) \subseteq \bigcup_{r \in R_{N_2}} B(r, 2q^{-N_2}).$$

Using $q > 28$ we obtain for $1 \leq \ell, k \leq n, \ell \neq k$

$$\begin{aligned} d(s_\ell, s_k) &\geq d(t_\ell, t_k) - d(t_\ell, s_\ell) - d(t_k, s_k) \\ &> q^{-j-1} - 2q^{-N_2} - 2q^{-N_2} \\ &\geq q^{-j-1} - 4q^{-j-2} \\ &> 4 \cdot 6q^{-j-2}. \end{aligned}$$

Hence we get by theorem 3.1.2

$$\begin{aligned}\mathbb{E}\sup_{r\in R_{N_2}} X_r &\geq 4q^{-j-2}\sqrt{\ln n}+\min_{1\leq \ell\leq n}\mathbb{E}\sup_{r\in R_{N_2}\cap B(s_\ell,6q^{-j-2})} X_r\\ &\geq 4q^{-j-2}\sqrt{\ln n}+\min_{1\leq \ell\leq n}\mathbb{E}\sup_{r\in R_{N_2}\cap B(t_\ell,4q^{-j-2})} X_r,\end{aligned}$$

the latter because $B(t_\ell,4q^{-j-2})\subseteq B(s_\ell,4q^{-j-2}+2q^{-N_2})\subseteq B(s_\ell,6q^{-j-2})$.

To prove that

$$\phi_{j+2}(t_\ell)\leq \mathbb{E}\sup_{r\in R_{N_2}\cap B(t_\ell,4q^{-j-2})} X_r$$

it is sufficient to show that $R_{N_2}\cap B(t_\ell,4q^{-j-2})\in \mathcal{C}_{N_2}(t_\ell,j+2)$:

Let $s\in B(t_\ell,2q^{-j-2})$.

We have to find $r\in R_{N_2}\cap B(t_\ell,4q^{-j-2})$ with $s\in B(r,2q^{-N_2})$.

Obviously,

$$d(s,t)\leq d(s,t_\ell)+d(t_\ell,t)\leq 2q^{-j-2}+q^{-j}\leq 2q^{-j}$$

and hence we get by definition of $\mathcal{C}_{N_2}(t,j)$ that $s\in B(r,2q^{-N_2})$ for some $r\in R_{N_2}$. Furthermore

$$d(r,t_\ell)\leq d(r,s)+d(s,t_\ell)\leq 2q^{-N_2}+2q^{-j-2}\leq 4q^{-j-2}$$

and thus $r\in R_{N_2}\cap B(t_\ell,4q^{-j-2})$.

We therefore obtain

$$\mathbb{E}\sup_{r\in R_{N_2}} X_r\geq 4q^{-j-2}\sqrt{\ln n}+\min_{1\leq \ell\leq n}\phi_{j+2}(t_\ell).$$

Note that $R_{N_2}\in\mathcal{C}_{N_2}(t,j)$ was arbitrary. Thus

$$\phi_j(t)\geq 4q^{-j-2}\sqrt{\ln n}+\min_{1\leq \ell\leq n}\phi_{j+2}(t_\ell).$$

We apply theorem 2.3.2 and get

$$\Theta_{N_1+1}^{N_2-1}\leq c\left(CF_{N_1}^{N_2}(T)+q^{-N_1-1}+q^{-N_1-1}\sqrt{\ln N(T,d,q^{-N_1-1})}\right).$$

Using $N(T,d,q^{-N_1-1})\geq N(T,d,q^{-N_0-1})\geq 2$ this implies the desired result.

□

3.3 Relations between $CF_{N_1}^{N_2}$ and $DF_{N_1}^{N_2}$

Set

$$\mathcal{D}_{N_2} = \left\{ S_{N_2} \subseteq T\,;\ \bigwedge_{\substack{s,t \in S_{N_2} \\ s \neq t}} \big(d(s,t) > 2q^{-N_2} \big) \right\}.$$

Note that $\mathcal{D}_\infty = \{S\,;\ S \subseteq T\}$.

Define

$$DF_{N_1}^{N_2}(T) = \sup \left\{ \sup_{t \in S_{N_2}} \mathbb{E} \sup_{\substack{s \in S_{N_2} \\ d(s,t) \le 2q^{-N_1}}} X_s\,;\ S_{N_2} \in \mathcal{D}_{N_2} \right\}$$

and

$$\overline{DF}_{N_1}^{N_2}(T) = \sup \left\{ \mathbb{E} \sup_{\substack{s,t \in S_{N_2} \\ d(s,t) \le 2q^{-N_1}}} |X_s - X_t|\,;\ S_{N_2} \in \mathcal{D}_{N_2} \right\}.$$

Proposition 3.3 Let $q > 2$. Then

$$CF_{N_1}^{N_2}(T) \le DF_{N_1}^{N_2}(T) \le \frac{q}{q-2} \cdot CF_{N_1 - 1}^{N_2 + 1}(T)$$

and

$$\overline{CF}_{N_1}^{N_2}(T) \le \overline{DF}_{N_1}^{N_2}(T) \le \frac{q}{q-2} \cdot \overline{CF}_{N_1 - 1}^{N_2 + 1}(T).$$

Proof: If $S_{N_2} \in \mathcal{D}_{N_2}$, there are two alternative cases:

either $S_{N_2} \in \mathcal{C}_{N_2}$ or we can find $t \in T \setminus S_{N_2}$ with $S_{N_2} \cup \{t\} \in \mathcal{D}_{N_2}$. Consequently, we can find $R_{N_2} \supseteq S_{N_2}$ with $R_{N_2} \in \mathcal{D}_{N_2} \cap \mathcal{C}_{N_2}$.

It holds

$$\sup_{t \in S_{N_2}} \mathbb{E} \sup_{\substack{s \in S_{N_2} \\ d(s,t) \le 2q^{-N_1}}} X_s \le \sup_{t \in R_{N_2}} \mathbb{E} \sup_{\substack{s \in R_{N_2} \\ d(s,t) \le 2q^{-N_1}}} X_s$$

and

$$\mathbb{E} \sup_{\substack{s,t \in S_{N_2} \\ d(s,t) \le 2q^{-N_1}}} |X_s - X_t| \le \mathbb{E} \sup_{\substack{s,t \in R_{N_2} \\ d(s,t) \le 2q^{-N_1}}} |X_s - X_t|.$$

Hence

$$DF_{N_1}^{N_2}(T) = \sup\left\{ \sup_{t\in R_{N_2}} \mathbb{E} \sup_{\substack{s\in R_{N_2}\\ d(s,t)\le 2q^{-N_1}}} X_s \, ; \; R_{N_2} \in \mathcal{D}_{N_2} \cap \mathcal{C}_{N_2} \right\}$$

and

$$\overline{DF}_{N_1}^{N_2}(T) = \sup\left\{ \mathbb{E} \sup_{\substack{s,t\in R_{N_2}\\ d(s,t)\le 2q^{-N_1}}} |X_s - X_t| \, ; \; R_{N_2} \in \mathcal{D}_{N_2} \cap \mathcal{C}_{N_2} \right\}.$$

Thus we get both left hand sides by the definitions of $CF_{N_1}^{N_2}$ and $\overline{CF}_{N_1}^{N_2}$. For the right hand sides let $T_{N_2+1} \in \mathcal{C}_{N_2+1}$ and $R_{N_2} \in \mathcal{D}_{N_2} \cap \mathcal{C}_{N_2}$ be arbitrary. We have to show that

$$\sup_{\tilde{s}\in R_{N_2}} \mathbb{E} \sup_{\substack{s\in R_{N_2}\\ d(s,\tilde{s})\le 2q^{-N_1}}} X_s \le \frac{q}{q-2} \cdot \sup_{\tilde{t}\in T_{N_2+1}} \mathbb{E} \sup_{\substack{t\in T_{N_2+1}\\ d(t,\tilde{t})\le 2q^{-N_1+1}}} X_t$$

and

$$\mathbb{E} \sup_{\substack{s,\tilde{s}\in R_{N_2}\\ d(s,\tilde{s})\le 2q^{-N_1}}} |X_s - X_{\tilde{s}}| \le \frac{q}{q-2} \cdot \mathbb{E} \sup_{\substack{t,\tilde{t}\in T_{N_2+1}\\ d(t,\tilde{t})\le 2q^{-N_1+1}}} |X_t - X_{\tilde{t}}|.$$

For that purpose we choose for any $s \in R_{N_2}$ some point $t(s) \in T_{N_2+1}$ with $s \in B(t(s), 2q^{-N_2-1})$. If $s, \tilde{s} \in R_{N_2}$, $s \neq \tilde{s}$, we get

$$d(t(s), t(\tilde{s})) \ge d(s,\tilde{s}) - d(s,t(s)) - d(\tilde{s}, t(\tilde{s})) \ge \left(1 - \frac{2}{q}\right) d(s,\tilde{s})$$

and

$$d(t(s), t(\tilde{s})) \le d(s,\tilde{s}) + d(s,t(s)) + d(\tilde{s}, t(\tilde{s})) \le \left(1 + \frac{2}{q}\right) d(s,\tilde{s}).$$

By theorem 3.1.3 and $1 + \frac{2}{q} \le q$ we conclude for $\tilde{s} \in R_{N_2}$

$$\mathbb{E} \sup_{\substack{s\in R_{N_2}\\ d(s,\tilde{s})\le 2q^{-N_1}}} X_s \le \frac{q}{q-2} \cdot \mathbb{E} \sup_{\substack{s\in R_{N_2}\\ d(s,\tilde{s})\le 2q^{-N_1}}} X_{t(s)} \le \frac{q}{q-2} \cdot \sup_{\tilde{t}\in T_{N_2+1}} \mathbb{E} \sup_{\substack{t\in T_{N_2+1}\\ d(t,\tilde{t})\le 2q^{-N_1+1}}} X_t$$

and

$$\begin{aligned}\mathbb{E} \sup_{\substack{s,\tilde{s}\in R_{N_2} \\ d(s,\tilde{s})\le 2q^{-N_1}}} |X_s - X_{\tilde{s}}| &\le \frac{q}{q-2}\cdot \mathbb{E} \sup_{\substack{s,\tilde{s}\in R_{N_2} \\ d(s,\tilde{s})\le 2q^{-N_1}}} |X_{t(s)} - X_{t(\tilde{s})}| \\ &\le \frac{q}{q-2}\cdot \mathbb{E} \sup_{\substack{t,\tilde{t}\in T_{N_2+1} \\ d(t,\tilde{t})\le 2q^{-N_1+1}}} |X_t - X_{\tilde{t}}|\end{aligned}$$

as desired. □

3.4 Relations between $\Theta_{N_1}^{N_2}, M_{N_1}^{N_2}, F_{N_1}^{N_2}$ and $CF_{N_1}^{N_2}$

Recall section 2.3.1.

Set for $T_{N_2} \in \mathcal{C}_{N_2}$ and $u \in SF(N_1)$

$$\mathcal{S}(M, T_{N_2}, u) = \left\{ (s_n); \quad \begin{array}{l} \bigwedge_n \left(s_n \in \ell_2, \|s_n\|_2 \le \frac{M}{\sqrt{\ln \max\{n,2\}}} \right), \\ \bigwedge_{t\in T_{N_2}} \left(t - u(t) \in \overline{\mathrm{conv}}\{0, s_1, s_2, \ldots\} \right) \end{array} \right\}.$$

Note that (s_n) may be a finite sequence.

We define

$$M_{N_1}^{N_2}(T) = \inf_{T_{N_2}\in\mathcal{C}_{N_2}} \sup_{u\in SF(N_1)} \inf \{M;\, \mathcal{S}(M, T_{N_2}, u) \neq \emptyset\}$$

and

$$F_{N_1}^{N_2}(T) = \inf_{T_{N_2}\in\mathcal{C}_{N_2}} \sup_{u\in SF(N_1)} \mathbb{E} \sup_{t\in T_{N_2}} |X_t - X_{u(t)}|.$$

Proposition 3.4 Let $N_1 \ge N_0$ and $q > 2$. Then

$$CF_{N_1+1}^{N_2}(T) \le 4F_{N_1}^{N_2}(T) \le c_1 M_{N_1}^{N_2}(T) \le c_2 \Theta_{N_1}^{N_2}(T, d).$$

Proof: We prove first that $M_{N_1}^{N_2}(T) \le c\Theta_{N_1}^{N_2}(T, d)$:

We take $\mathcal{A} \in \mathcal{Z}_{N_1}^{N_2}(T, d)$ and $\omega \in \mathcal{G}(\mathcal{A})$ with

$$\Theta_{N_1,\mathcal{A},\omega}^{N_2} \le 2\Theta_{N_1}^{N_2}(T, d).$$

If $N_2 < \infty$ then fix some point $r_A \in A$ for any $A \in \mathcal{A}_{N_2}$ and denote the set $\{r_A;\ A \in \mathcal{A}_{N_2}\}$ by T_{N_2}. If $N_2 = \infty$, set $T_{N_2} = T$. Obviously, $T_{N_2} \in \mathcal{C}_{N_2}$ and thus

$$M_{N_1}^{N_2}(T) \le \sup_{u \in SF(N_1)} \inf \{M;\ \mathcal{S}(M, T_{N_2}, u) \neq \emptyset\}.$$

We choose $u_0 \in SF(N_1)$ with

$$\frac{1}{2} \sup_{u \in SF(N_1)} \inf \{M;\ \mathcal{S}(M, T_{N_2}, u) \neq \emptyset\} \le \inf \{M;\ \mathcal{S}(M, T_{N_2}, u_0) \neq \emptyset\}.$$

By definition of $SF(N_1)$ we find some $\widetilde{\mathcal{A}}_{N_1} \in SP(N_1)$ with $u_0 \in SF(\widetilde{\mathcal{A}}_{N_1})$. Fix $\widetilde{\omega}_{N_1} \in S\mathcal{G}(\widetilde{\mathcal{A}}_{N_1})$ and recall the proof of proposition 2.3.1. We get

$$\Theta_{N_1, \widetilde{\mathcal{A}}, \widetilde{\omega}}^{N_2} \le \left(1 + \sqrt{2}\frac{q}{q-1}\right) \Theta_{N_1, \mathcal{A}, \omega}^{N_2},$$

where for $N_1 < j \le N_2$

$$\widetilde{\mathcal{A}}_j = \{A \cap B;\ A \cap B \neq \emptyset, A \in \mathcal{A}_j, B \in \widetilde{\mathcal{A}}_{N_1}\}$$

and

$$\widetilde{\omega}_j(A \cap B) = \omega_j(A) \cdot (N(T, d, q^{-N_1}) + 1)^{-1}.$$

Now we obtain by using $N_1 \ge N_0$ and proposition 2.1.3

$$\begin{aligned}
&\sup_{t \in T} \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln \frac{1}{2^{-j+N_1} \widetilde{\omega}_j\left(\widetilde{A}_j(t)\right)}} \\
\le\ & \Theta_{N_1, \widetilde{\mathcal{A}}, \widetilde{\omega}}^{N_2} + \sum_{j=N_1+1}^{N_2} q^{-j} \sqrt{\ln 2^{j-N_1}} \\
\le\ & \Theta_{N_1, \widetilde{\mathcal{A}}, \widetilde{\omega}}^{N_2} + q \sum_{j=1}^{\infty} q^{-j} \sqrt{j} \cdot q^{-N_1-1} \sqrt{\ln N(T, d, q^{-N_1-1})} \\
\le\ & \left(1 + q \sum_{j=1}^{\infty} q^{-j} \sqrt{j}\right) \Theta_{N_1, \widetilde{\mathcal{A}}, \widetilde{\omega}}^{N_2} \\
\le\ & c_0 \Theta_{N_1}^{N_2}(T, d),
\end{aligned}$$

where

$$c_0 = \left(1 + q\sum_{j=1}^{\infty} q^{-j}\sqrt{j}\right) \cdot \left(1 + \sqrt{2}\frac{q}{q-1}\right) \cdot 2.$$

Setting

$$a_j(A) = \frac{1}{c_0\Theta_{N_1}^{N_2}(T,d)} q^{-j}\sqrt{\ln\frac{1}{2^{-j+N_1}\widetilde{\omega}_j(A)}}$$

for $A \in \widetilde{\mathcal{A}}_j$ we get for $t \in T$

$$\sum_{j=N_1+1}^{N_2} a_j\left(\widetilde{A}_j(t)\right) \leq 1.$$

We choose a point $r_A \in A$ for any $A \in \widetilde{\mathcal{A}}_j, N_1 < j \leq N_2$, in case of $j = N_2$ (if $N_2 < \infty$) in such a way that $\{r_A;\ A \in \widetilde{\mathcal{A}}_{N_2}\} \supseteq T_{N_2}$.
For $A \in \widetilde{\mathcal{A}}_{N_1}$, we denote the point $u_0(A) \in T$ with

$$u_0(t) = u_0(A),\ t \in A, \text{ and } A \subseteq B\left(u_0(A), q^{-N_1}\right)$$

by r_A (see section 2.3.1). Set

$$s(A) = \frac{1}{a_j(A)}(r_A - r_{A'})$$

for $N_1 + 1 \leq j \leq N_2$ and $A \in \widetilde{\mathcal{A}}_j$, where A' is the set in $\widetilde{\mathcal{A}}_{j-1}$ with $A \subseteq A'$.
Consider the set $S = \{s(A);\ A \in \widetilde{\mathcal{A}}_j, N_1 + 1 \leq j \leq N_2\}$.
Note that for $t \in T_{N_2}$

$$\begin{aligned} t - u_0(t) &= \sum_{j=N_1+1}^{N_2} r_{\widetilde{A}_j(t)} - r_{\widetilde{A}_{j-1}(t)} \\ &= \sum_{j=N_1+1}^{N_2} a_j\left(\widetilde{A}_j(t)\right) s\left(\widetilde{A}_j(t)\right) \\ &\in \overline{\operatorname{conv}}\left(\{0\} \cup S\right). \end{aligned}$$

Next we take $A \in \widetilde{\mathcal{A}}_j$ and $A' \in \widetilde{\mathcal{A}}_{j-1}$ with $A \subseteq A'$, $N_1 + 1 \leq j \leq N_2$, and we get

$$\|s(A)\|_2 = \left\|\frac{1}{a_j(A)}(r_A - r_{A'})\right\|_2$$

$$\begin{aligned} &\leq \frac{c_0\Theta_{N_1}^{N_2}(T,d)\cdot 2q^{-j+1}}{q^{-j}\sqrt{\ln\frac{1}{2^{-j+N_1}\widetilde{\omega}_j(A)}}} \\ &= \frac{2qc_0\Theta_{N_1}^{N_2}(T,d)}{\sqrt{\ln\frac{1}{2^{-j+N_1}\widetilde{\omega}_j(A)}}}. \end{aligned}$$

Furthermore,

$$\sum_{j=N_1+1}^{N_2}\sum_{A\in\widetilde{\mathcal{A}}_j} 2^{-j+N_1}\widetilde{\omega}_j(A)\leq 1.$$

Hence we can number S in such a way that $S=\{s_1,s_2,\ldots\}$ and the sequence

$$2^{-j(n)+N_1}\widetilde{\omega}_{j(n)}(A^n)$$

with $j(n)$ and A^n determined by

$$s_n=s(A^n) \text{ and } A^n\in\widetilde{\mathcal{A}}_{j(n)}$$

is decreasing. Obviously,

$$2^{-j(n)+N_1}\widetilde{\omega}_{j(n)}(A^n)\leq\frac{1}{n}$$

and

$$2^{-j(1)+N_1}\widetilde{\omega}_{j(1)}(A^1)\leq\frac{1}{2},$$

hence

$$\|s_n\|_2\leq\frac{2qc_0\Theta_{N_1}^{N_2}(T,d)}{\sqrt{\ln\max\{n,2\}}}.$$

If follows that $\mathcal{S}\left(2qc_0\Theta_{N_1}^{N_2}(T,d),T_{N_2},u_0\right)\neq\emptyset$ and thus

$$\frac{1}{2}M_{N_1}^{N_2}(T)\leq\inf\{M\,;\ \mathcal{S}(M,T_{N_2},u_0)\neq\emptyset\}\leq 2qc_0\Theta_{N_1}^{N_2}(T,d).$$

Now we prove that $F_{N_1}^{N_2}(T)\leq cM_{N_1}^{N_2}(T)$:

For that purpose we choose $T_{N_2}\in\mathcal{C}_{N_2}$ such that

$$2M_{N_1}^{N_2}(T)\geq\sup_{u\in SF(N_1)}\inf\{M;\ \mathcal{S}(M,T_{N_2},u)\neq\emptyset\}.$$

Of course,

$$F_{N_1}^{N_2}(T) \leq \sup_{u \in SF(N_1)} \mathbb{E} \sup_{t \in T_{N_2}} |X_t - X_{u(t)}|.$$

Take $u_0 \in SF(N_1)$ with

$$\frac{1}{2} \sup_{u \in SF(N_1)} \mathbb{E} \sup_{t \in T_{N_2}} |X_t - X_{u(t)}| \leq \mathbb{E} \sup_{t \in T_{N_2}} |X_t - X_{u_0(t)}|.$$

We get

$$2M_{N_1}^{N_2}(T) \geq \inf\{M;\ \mathcal{S}(M, T_{N_2}, u_0) \neq \emptyset\}$$

and hence $\mathcal{S}\left(3M_{N_1}^{N_2}(T), T_{N_2}, u_0\right) \neq \emptyset$.

Thus we find (s_n) with

$$\|s_n\|_2 \leq \frac{3M_{N_1}^{N_2}(T)}{\sqrt{\ln \max\{n, 2\}}} \quad \text{and} \quad \bigwedge_{t \in T_{N_2}} \left(t - u_0(t) \in \overline{\text{conv}}\{0, s_1, s_2, \ldots\}\right).$$

By using theorem 3.1.4 we obtain

$$\begin{aligned} \frac{1}{2} F_{N_1}^{N_2}(T) &\leq \mathbb{E} \sup_{t \in T_{N_2}} |X_t - X_{u_0(t)}| \\ &\leq \mathbb{E} \sup_n |X_{s_n}| \\ &\leq cM_{N_1}^{N_2}(T). \end{aligned}$$

It remains to prove that $CF_{N_1+1}^{N_2}(T) \leq 4F_{N_1}^{N_2}(T)$:

We choose $T_{N_2} \in \mathcal{C}_{N_2}$ with

$$2F_{N_1}^{N_2}(T) \geq \sup_{u \in SF(N_1)} \mathbb{E} \sup_{t \in T_{N_2}} |X_t - X_{u(t)}|.$$

It follows $(q > 2)$

$$CF_{N_1+1}^{N_2}(T) \leq \sup_{t \in T_{N_2}} \mathbb{E} \sup_{\substack{s \in T_{N_2} \\ d(s,t) \leq 2q^{-N_1-1}}} X_s \leq \sup_{t \in T_{N_2}} \mathbb{E} \sup_{\substack{s \in T_{N_2} \\ d(s,t) \leq q^{-N_1}}} X_s.$$

Take now $t_0 \in T_{N_2}$ such that

$$\frac{1}{2} \sup_{t \in T_{N_2}} \mathbb{E} \sup_{\substack{s \in T_{N_2} \\ d(s,t) \leq q^{-N_1}}} X_s \leq \mathbb{E} \sup_{\substack{s \in T_{N_2} \\ d(s,t_0) \leq q^{-N_1}}} X_s.$$

Recall section 2.3.1 and choose a cover of T by using at most $N(T, d, q^{-N_1})+1$ balls of radius q^{-N_1}, which contains $B(t_0, q^{-N_1})$. We take a partition $\mathcal{A}_{N_1}$ of T which is induced by this cover and which satisfies $B(t_0, q^{-N_1}) \in \mathcal{A}_{N_1}$.

Set $u_0(s) = t_0$ for $s \in B(t_0, q^{-N_1})$ and $u_0(s) = t_A$ for $s \in A$, where $A \in \mathcal{A}_{N_1}$, $A \neq B(t_0, q^{-N_1})$ and t_A is a point with $A \subseteq B(t_A, q^{-N_1})$. Hence $u_0 \in SF(N_1)$ and

$$\begin{aligned}
\mathbb{E} \sup_{\substack{s \in T_{N_2} \\ d(s,t_0) \leq q^{-N_1}}} X_s &= \mathbb{E} \sup_{\substack{s \in T_{N_2} \\ s \in B(t_0, q^{-N_1})}} (X_s - X_{t_0}) \\
&\leq \mathbb{E} \sup_{s \in T_{N_2}} |X_s - X_{u_0(s)}| \\
&\leq \sup_{u \in SF(N_1)} \mathbb{E} \sup_{t \in T_{N_2}} |X_t - X_{u(t)}|.
\end{aligned}$$

Therefore we obtain

$$CF^{N_2}_{N_1+1}(T) \leq 4F^{N_2}_{N_1}(T).$$

$\square$

3.5 Convex-Hull-Problem

Note that we get for $T_{N_2} \in \mathcal{C}_{N_2}(T)$ by definition

$$T \subseteq \bigcup_{r \in T_{N_2}} B(r, 2q^{-N_2})$$

and hence

$$\mathrm{conv}T \subseteq \bigcup_{r \in \mathrm{conv}T_{N_2}} B(r, 2q^{-N_2}).$$

Consequently,

$$\{\mathrm{conv}T_{N_2};\ T_{N_2} \in \mathcal{C}_{N_2}(T)\} \subseteq \mathcal{C}_{N_2}(\mathrm{conv}T).$$

By using $D(\mathrm{conv}T) = D(T) \leq 2q^{-N_0}$ we obtain

$$\begin{aligned} CF_{N_0}^{N_2}(\mathrm{conv}T) &= \inf\left\{\mathbb{E}\sup_{s\in R_{N_2}} X_s;\ R_{N_2} \in \mathcal{C}_{N_2}(\mathrm{conv}T)\right\} \\ &\leq \inf\left\{\mathbb{E}\sup_{s\in R_{N_2}} X_s;\ R_{N_2} = \mathrm{conv}T_{N_2}, T_{N_2} \in \mathcal{C}_{N_2}(T)\right\} \\ &= \inf\left\{\mathbb{E}\sup_{s\in T_{N_2}} X_s;\ T_{N_2} \in \mathcal{C}_{N_2}(T)\right\} \\ &= CF_{N_0}^{N_2}(T). \end{aligned}$$

Now we are able to formulate the convex-hull-problem:

The understanding of the inequality

$$CF_{N_0}^{N_2}(\mathrm{conv}T) \leq CF_{N_0}^{N_2}(T)$$

from a geometric point of view (with help of majorizing measures).

Note that this problem (for $N_2 = \infty$) has been open for a long time, indicating that it is hard to solve indeed. However, it remains interesting for an understanding of majorizing measures. Therefore, further studies are needed.

Bibliography

[Ad] R. J. Adler. *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes.* IMS Lecture Notes - Monograph Series, 12, Hayward, California (1990).

[Du] R. M. Dudley. The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. J. Funct. Anal. 1 (1967), 290–330.

[F1] X. Fernique. Réguarité des trajectoires des fonctions aléatoires gaussiennes. École d'Été de Probabilités de St-Flour 1974, Lecture Notes in Math. 480, Springer (1975), 1–96.

[F2] X. Fernique. *Fonctions aléatoires gaussiennes vecteurs aléatoires gaussiens.* Les Publications CRM, Montreal (1997).

[Ka] J.-P. Kahane. Une inégalité du type de Slepian et Gordon sur les processus gaussiens. Israel J. Math. 55 (1986), 109–110.

[Le] M. Ledoux. Isoperimetry and Gaussian Analysis. Lectures on Probability Theory and Statistics, Lecture Notes in Math. 1648, Springer (1996), 165–294.

[Lf] M. A. Lifshits. *Gaussian Random Functions.* Kluwer Academic Publishers (1995).

[LL] W. V. Li, W. Linde. Metric Entropy of Convex Hulls in Hilbert spaces via Majorizing Measures. Preprint (1998).

[LT] M. Ledoux, M. Talagrand. *Probability in Banach spaces: Isoperimetry and processes.* Springer (1991).

[Su] V. N. Sudakov. A remark on the criterion of continuity of Gaussian sample functions. Proceedings of the Second Japan-USSR Symposium on Probability Theory, Lecture Notes in Math. 330, Springer (1973), 444–454.

[T1] M. Talagrand. Regularity of Gaussian processes. Acta Math.159 (1987), 99–149.

[T2] M. Talagrand. Majorizing measures: The generic chaining. Ann. Probab. 24 (1996), 1049–1103.

[T3] M. Talagrand. The supremum of some canonical processes. Amer. J. Math. 116 (1994), 284–325.

[T4] M. Talagrand. Constructions of majorizing measures, Bernoulli processes and cotype. Geometric and Functional Analysis 4 (1994), 660–717.

[VW] A. W. van der Vaart, J. A. Wellner. *Weak Convergence and Empirical Processes.* Springer (1996).